MODERN GEOMETRY

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MODERN GEOMETRY

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PREFACE

THE present volume is a sequel to the Elementary Geometry written by the same authors.

It covers the schedule of Modern Plane Geometry required for the Special Examination in Mathematics for the Ordinary B.A. Degree at Cambridge; and represents what we take to be a useful course for any student of Mathematics, whether he intends to read for Mathematical Honours, or to take up Physics or Engineering. For those who ultimately make a special study of Geometry, this book would serve as an introduction to more advanced treatises.

Our experience tends to shew that boys find no little difficulty, at the outset, in mastering the ideas of Modern Plane Geometry; and, especially, in solving the problems usually set. We have therefore put in a number of quite easy exercises, arranged to lead by easy stages to the Scholarship questions at the end of Chapters.

We have to thank Mr H. M. Taylor for permission to use some of the exercises included in his edition of Euclid.

C. G.
A. W. S.

June, 1908.

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CHAPTER I.

THE SENSE OF A LINE.

1. THROUGHOUT this book the word 'line' will generally be used in the sense of 'straight line.'

2. In elementary Geometry, the notation AB as applied to a straight line has one of two meanings:—

(1) The unlimited straight line defined by, and passing through, the points A, B.

(2) The limited segment of that line that lies between A and B.

It is now necessary to explain a third use of the notation.

We may wish to discriminate between the *step* from A to B, and the *step* from B to A. In fact, we may regard AB and BA as different, AB meaning the step from A to B, and BA the step from B to A; AB and BA being **in different senses.** If this idea is present, it is very usual to draw attention to the fact by writing a bar over the letters: thus, $\overline{AB}$ means **the step from A to B.**

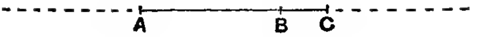

fig. 1.

3. Reverting for a moment to the more elementary idea, we see that in fig. 1

$$AB + BC = AC,$$

and we may interpret this as meaning that the consecutive steps from A to B, and from B to C, are together equivalent to the single step from A to C.

It is a natural extension of this if we agree to say that in fig. 2

fig. 2.

$$\overline{AB} + \overline{BC} = \overline{AC} \quad \text{..........................} \quad \text{(i)},$$

meaning that, if we step in succession from A to B, and from B to C, the result is the same as if we had stepped at once from A to C.

The above is an **extension of the idea of addition.** The statement (i) is, in fact, to be regarded as true for all cases, and as following directly from the extended idea of addition.

4. As a particular case of the above

$$\overline{AB} + \overline{BA} = 0,$$

$$\therefore\ \overline{BA} = -\overline{AB} \quad\text{.......................(ii).}$$

If, then, we agree to regard as positive all steps measured in one sense, we may regard as negative all steps measured in the opposite sense.

5. Let A, B, C, D, E be any number of points, *arranged in any order* along a line.

It follows from the extended idea of addition that

$$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} = \overline{AE}.$$

$$\text{But } \overline{AE} = -\overline{EA},$$

$$\therefore\ \overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EA} = 0 \quad\text{...........(iii).}$$

6. Suppose that O is an origin and A, B any two points whatever in a line.

fig. 3.

Then $$\overline{OA} + \overline{AB} = \overline{OB},$$

$$\therefore\ \overline{AB} = \overline{OB} - \overline{OA} \quad\text{.....................(iv).}$$

7. Let C be the mid-point of AB.

Then $$\overline{OA} + \overline{AC} = \overline{OC},$$

$$\overline{OB} + \overline{BC} = \overline{OC}.$$

But $$\overline{BC} = -\overline{AC},$$

$$\therefore\ \text{adding, } \overline{OA} + \overline{OB} = 2\overline{OC},$$

$$\therefore\ \overline{OC} = \frac{\overline{OA} + \overline{OB}}{2} \quad\text{....................(v).}$$

The above results, i—v, are very important and useful: their value lies in this, that they may be employed with confidence without any reference to the figure; they are true whatever be the order of the points concerned.

Ex. 1. Verify the truth of the above results, i–v, by taking numerical instances and placing the points in various orders.

Ex. 2. A, B, C, D are points in any order on a straight line. Prove that

$$\overline{AB}\,.\,\overline{CD}+\overline{AC}\,.\,\overline{DB}+\overline{AD}\,.\,\overline{BC}=0.$$

Verify by taking numerical instances.

Ex. 3. If AB be divided in C so that $m\,.\,\overline{AC}=n\,.\,\overline{CB}$, and if O be any point on the infinite line AB,

$$\overline{OC}=\frac{m\,.\,\overline{OA}+n\,.\,\overline{OB}}{m+n}.$$

Ex. 4. If O, A, B, C be points on a line; and if P, Q, R be the mid-points of BC, CA, AB respectively, then

$$\overline{OP}\,.\,\overline{BC}+\overline{OQ}\,.\,\overline{CA}+\overline{OR}\,.\,\overline{AB}=0.$$

Ex. 5. If A, B, C, D be points on a line, and

$$\frac{\overline{AC}\,.\,\overline{DB}}{\overline{CB}\,.\,\overline{AD}}=\lambda,$$

then

$$\frac{\overline{AB}\,.\,\overline{DC}}{\overline{BC}\,.\,\overline{AD}}=1-\lambda.$$

Ex. 6. If A, B, X, Y are four collinear points, and P is a point on the same straight line such that PA . PB = PX . PY, show that

$$PA\,.\,BX\,.\,BY=PB\,.\,AX\,.\,AY.$$

8. Before leaving the subject of the 'sign' or 'sense' of segment of a line, one more remark must be made.

If there be two lines inclined to one another, it is not possible to compare, as regards sign, segments of the one line with segments of the other line. In fact, before any such comparison is possible we must add $\sqrt{-1}$ to the stock of symbols we command.

The Sense of an Angle.

9. There is a certain analogy (which will be developed later) between

(*a*) a point, lying on a certain line, and moving along it, and

(*b*) a line, passing through a certain point, and rotating round it.

Just as in case (*a*) we regarded motion in one sense as positive and motion in the opposite sense as negative, so in case (*b*) we may regard rotation in the one sense as positive and rotation in the opposite sense as negative.

Thus, if an angle AOB is looked upon as having been swept out by a radius rotating from OA to OB, we may call it positive; while, if it is looked at as having been swept out by a radius rotating from OB to OA, we should call it negative.

When it is convenient to use this idea, we should say that $\angle AOB = -\angle BOA$.

CHAPTER II.

INFINITY.

1. THERE is one exception to the rule that two coplanar straight lines define a point by their intersection.

This is the case of two parallel straight lines.

There is one exception to the rule that three points define one circle passing through them.

This is the case of three collinear points.

There is one exception to the rule that a finite straight line may be divided both internally and externally in a given ratio.

This is the case of the ratio of equality.

These and other exceptions can be removed by means of the mathematical fiction called 'infinity.'

It will be seen later on that, by means of the concept 'infinity' we are able to state as true *without exception* an indefinite number of results which would otherwise have to be stated in a limited form.

2. Point at infinity on a straight line.

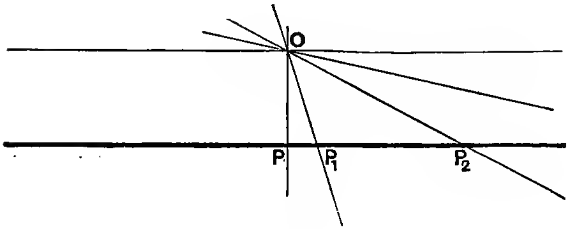

fig. 4.

Let a straight line, always passing through O, start from the position OP and revolve in a *counter-clockwise* direction, until it becomes parallel to the straight line PP_1.

In each of its positions, **the revolving line cuts the line PP_1 in a single point,** until the revolving line becomes parallel to PP_1.

When this happens, the statement in black type suddenly ceases to be true.

The more nearly the revolving line approaches to the parallel, the more distant does the point of intersection become.

It is found to be convenient to say that the revolving line, when parallel to PP_1, still cuts it; namely, in a **point at infinity** on PP_1. It will be seen below that these 'mathematical fictions' —points at infinity—possess many properties in common with ordinary points.

If the revolving line starts afresh from OP and now revolves in the *clockwise* direction, it might be supposed that, when parallel to PP_1, it determines another point at infinity on PP_1.

We shall find, however, that it leads to simpler statements if we agree to say that this point at infinity is identical with that obtained formerly.

The reader may object that this is an unreasonable convention, in that it allows a 'point at infinity' to be infinitely distant from itself.

In answer to this objection, it must be explained that we have not stated that points at infinity enjoy *all* the properties of ordinary points.

3. As an illustration of the uniformity of statement obtained by the conventions already explained, the following are now given as true without exception.

(i) Any two coplanar straight lines define one point by their intersection.

(ii) Two straight lines cannot enclose a space. (If we had agreed to admit *two* points at infinity on a straight line, two parallel straight lines would define *two* points and would enclose an infinite space.)

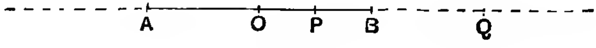

fig. 5.

(iii) If it is required to divide AB in a given ratio, so that, say, $\frac{AP}{PB} > 1$, the problem admits of two solutions: either by internal division (P) or by external division (Q).

If the ratio is gradually altered so that it approaches unity, P will approach the middle point O, and Q will move off indefinitely to the right.

When the ratio becomes unity, the internal point of division is O, and the external point of division is the point at infinity on AB.

If the ratio had approached unity from below instead of from above, the internal point of division would have approached O from the left; and the external point of division would have moved off indefinitely to the left till, in the limit, it coincided with the point at infinity, as before.

4. A set of parallel lines cointersect in one point at infinity, namely *the* point at infinity belonging to that set. In fact, a set of parallel lines is a particular case of a set (or pencil) of concurrent lines.

To each set of parallels in a plane, in other words to each *direction*, there belongs a point at infinity. If we consider all possible directions, we have a singly infinite set of points at infinity; and it may be asked what is the locus of these points.

The locus, apparently, has this property; that every straight line in the plane cuts it in one point. For a straight line cuts the locus in *the* point at infinity on that straight line.

In virtue of the above property, the locus must, itself, be regarded as a straight line. To call it anything else, e.g. a circle, would introduce inconsistency of language; and the whole object of introducing points at infinity is to make mathematical language consistent.

The locus of all points at infinity in a plane is, accordingly, **the line at infinity** in the plane.

This line has many of the properties of ordinary lines, while it has other properties that are unfamiliar; e.g. it can be shown to make any angle whatsoever with itself.

5. Limit of a circle of infinite radius.

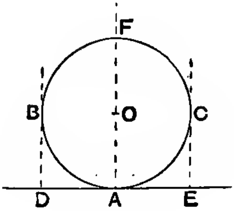

fig. 6.

Suppose that the circle in fig. 6 continually touches the line DAE in A, while the radius continually increases without limit, and the centre O recedes to infinity along AF produced.

The circle will flatten out, and the semicircle BAC will clearly tend to coincide with the infinite line DAE.

But it cannot be supposed that the limit of the circle is simply DAE; for a circle is cut by any line in 2 points, while DAE is cut by any line in 1 point: an essential distinction.

In fact, all the points on the semicircle BFC recede to infinity, and tend to lie on the line at infinity.

Therefore a circle of infinite radius with centre at infinity consists of **an infinite straight line together with the straight line at infinity.**

Ex. **7.** In the limit of figure 6 examine what becomes of the points C, B and of the tangents EC, DB.

Ex. **8.** Find what becomes of the theorem that 'a chord of a circle subtends equal or supplementary angles at all points of the circumference' for the case when the circle becomes a finite line *plus* the line at infinity.

CHAPTER III.

THE CENTROID.

THE properties of the **centroid** are mainly of interest in connection with statics, where they apply to the **centre of gravity** of a system of weights. The idea, however, is essentially geometrical; and will, therefore, be developed briefly in this place.

Definition. The **centroid** of n points in a plane, P_1, P_2, P_3, P_4, ... P_n is arrived at by the following construction. Bisect P_1P_2 in A. Divide AP_3 in B so that $2AB = BP_3$. Divide BP_4 in C so that $3BC = CP_4$; and so forth. The final point obtained by this process is G, the centroid of P_1 ... P_n.*

* The reader will notice that this definition is faulty, inasmuch that a doubt remains whether we should reach the same point G if we took the points P in a different order. It is proved below that the point G is uniqus.

Theorem 1.

If the distances of two points P, Q from a line XY be y_1, y_2 (sign being taken into account); and if G be taken on PQ so that $h.\mathrm{PG} = k.\mathrm{GQ}$, then the distance of G from the line is

$$\frac{hy_1 + ky_2}{h+k}.$$

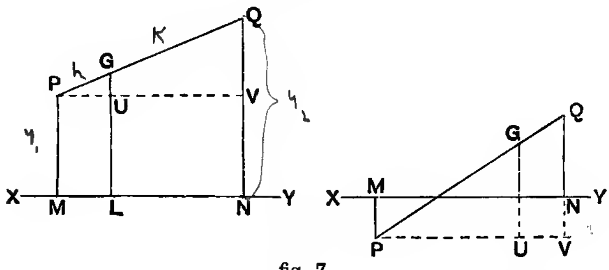

fig. 7.

Through P draw PUV ‖ to XY, meeting GL, QN (produced if necessary) in U, V.

Then, *in every case** (sign being taken into account)

$$\mathrm{UG} : \mathrm{VQ} = \mathrm{PG} : \mathrm{PG} + \mathrm{GQ}$$
$$= k : k + h,$$
$$\therefore \quad \mathrm{UG} = \frac{k}{h+k} . \mathrm{VQ}.$$

But $\mathrm{VQ} = \mathrm{NQ} - \mathrm{NV} = \mathrm{NQ} - \mathrm{MP}$

$$= y_2 - y_1,$$

and $\mathrm{LG} = \mathrm{UG} + \mathrm{MP}$

$$= \frac{k}{h+k}(y_2 - y_1) + y_1$$
$$= \frac{ky_2 - ky_1 + hy_1 + ky_1}{h+k}$$
$$= \frac{hy_1 + ky_2}{h+k}.$$

* This proof is a good instance of the fact explained in Chap. I., that the attribution of sign to lines makes us, in a measure, independent of the variety of figures that may be drawn.

THEOREM 2.

If the distances of points $P_1, \ldots, P_n$ from a line be $y_1, \ldots, y_n$ (sign being taken into account), the distance of the centroid G from the line is

$$\frac{y_1 + y_2 + \ldots + y_n}{n}.$$

The mid-point of P_1P_2 is A, and its distance is $\frac{y_1 + y_2}{2}$.

B is taken so that $2AB = BP_3$,

$$\therefore \text{ the distance of B is } \quad \frac{2\left(\frac{y_1 + y_2}{2}\right) + y_3}{2+1}$$

$$= \frac{y_1 + y_2 + y_3}{3}.$$

C is taken so that $3BC = CP_4$,

$$\therefore \text{ the distance of C is } \quad \frac{3\left(\frac{y_1 + y_2 + y_3}{3}\right) + y_4}{3+1}$$

$$= \frac{y_1 + y_2 + y_3 + y_4}{4},$$

etc., etc.

Finally the distance of G from the line is

$$\frac{y_1 + y_2 + y_3 \cdots + y_n}{n}.$$

THEOREM 3.

If the coordinates of P_1, P_2, ..., P_n, with respect to two axes at right angles, be (x_1, y_1) (x_2, y_2) (x_3, y_3) ... (x_n, y_n) the coordinates of the centroid are

$$\frac{x_1 + x_2 + \ldots + x_n}{n}, \quad \frac{y_1 + y_2 + \ldots + y_n}{n}.$$

This follows immediately from Theorem 2.

Theorem 3 makes it clear, from the symmetry of the expressions for the coordinates of G, that the same centroid would have been reached if the points P had been taken in any other order.

The centroid is therefore (i) unique, (ii) fixed relative to the points P.

From the fact that the same centroid is obtained in whatever order the points are taken, a class of geometrical theorems may be deduced of which the following is an example.

Example. **The medians of a triangle meet in a point, and each median is trisected at this point.**

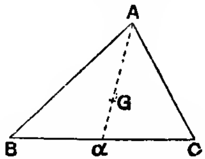

fig. 8.

Consider the centroid of the three points A, B, C. Let BC be bisected at α, and let G be taken on $A\alpha$ so that $AG = 2G\alpha$. Then G is the centroid: it lies on the median $A\alpha$ and trisects it.

Similarly the same point G lies on each of the other medians, and trisects it.

Hence the medians meet in a point, which trisects each median.

Ex. **9.** ABCD being a quadrilateral, the joins of the mid-points of AB, CD; of AC, BD; of AD, BC meet in a point; and each join is bisected at this point.

Ex. **10.** A, B, C, D are four points in a plane. Let the centroids of the triangles BCD, CDA, DAB, ABC be α, β, γ, δ respectively. Then Aα, Bβ, Cγ, Dδ meet in a point; and are divided in the same ratio at this point.

Ex. **11.** Assuming the existence of a centroid in three dimensions, generalise Exs. 9 and 10 for the case in which ABCD is a tetrahedron.

Ex. **12.** If G be the centroid of P_1, P_2, ... P_n, and GM_1, GM_2, GM_3, ... GM_n, be the projections of GP_1, GP_2, GP_3, ... GP_n on a line through G; then $\Sigma GM = 0$.

Ex. **13.** O being any point, and G the centroid of n points P_1, P_2, ... P_n,

$$\Sigma OP^2 = \Sigma GP^2 + n \, . \, OG^2.$$

(Use the extension of Pythagoras' theorem.)

CHAPTER IV.

THE TRIANGLE.

Notation. Special points and quantities will be denoted by the following letters, in the course of the present chapter.

A, B, C......vertices of the triangle,
D, E, F......feet of the altitudes,
α, β, γ ......mid-points of the sides,
X, Y, Z......points of contact of the in-circle,
a, b, c ......lengths of the sides,
s ....semi-perimeter ($2s = a + b + c$),
R ....circum-radius,
r ....in-radius,
r_1, r_2, r_3 ...ex-radii,
Δ ....area of the triangle,
S ....circumcentre,
H ....orthocentre,
G ....centroid,
I ....in-centre,
I_1, I_2, I_3 ...ex-centres,
N ....nine-points centre,
P, Q, R......mid-points of HA, HB, HC.

Theorem 4.

$$\Delta = \tfrac{1}{2}bc \sin \mathsf{A}.$$

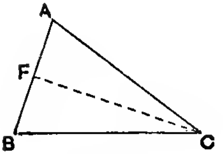

fig. 9.

Case I. *If* $\angle$ A *is acute.*

$$\text{Draw } \mathsf{CF} \perp \text{ to } \mathsf{AB}$$

$$\Delta = \tfrac{1}{2}\mathsf{AB}\,.\,\mathsf{CF}.$$

$$\text{But } \mathsf{CF} = \mathsf{CA} \sin \mathsf{A},$$

$$\therefore\ \Delta = \tfrac{1}{2}\mathsf{AB}\,.\,\mathsf{CA} \sin \mathsf{A}$$

$$= \tfrac{1}{2}bc \sin \mathsf{A}.$$

Similarly $\Delta = \tfrac{1}{2}ca \sin \mathsf{B} = \tfrac{1}{2}ab \sin \mathsf{C}.$

Case II. *If* $\angle$ A *is obtuse.*

The proof is left to the reader.

Ex. 14. Prove the above theorem for the case in which $\angle$ A is obtuse.

Ex. 15. Prove the theorem that the ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

Ex. 16. Two sides OP, OR of a variable parallelogram OPQR always lie along two fixed lines OX, OY; and Q describes the locus defined by OP . PQ = constant. Prove that the area of the parallelogram is constant. 2

OQ

Ex. 17. Deduce from Theorem 4 that

$$\frac{a}{\sin \mathsf{A}} = \frac{b}{\sin \mathsf{B}} = \frac{c}{\sin \mathsf{C}}.$$

Theorem 5.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

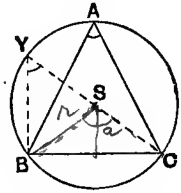

fig. 10.

Case i. *If the triangle is acute angled.*

Join CS.

Produce CS to meet circumcircle in Y.

Join BY.

Since CY is a diameter of the ⊙,

$$\therefore \angle CBY \text{ is a rt} \angle.$$

$$\text{Also } \angle BYC = \angle BAC,$$

$$\therefore \sin A = \sin Y = \frac{BC}{CY} = \frac{a}{2R},$$

$$\therefore \frac{a}{\sin A} = 2R.$$

$$\text{Similarly } \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Case ii. *If the triangle is obtuse angled.*

The proof of this case is left to the reader.

Ex. **18.** Prove Case ii of Theorem 5.

Ex. **19.** Prove that $$R=\frac{abc}{4\Delta}.$$

Ex. **20.** Prove that the circum-radius of an equilateral triangle of side x is approximately $\cdot 577x$.

Ex. **21.** SAP, PBQ, QCR, RDS are lines bisecting the exterior angles of a convex quadrilateral ABCD. Prove that

$$PB \,.\, QC \,.\, RD \,.\, SA = PA \,.\, SD \,.\, RC \,.\, QB.$$

Ex. **22.** Deduce from Theorem 5 the fact that the bisector of the vertical angle of a triangle divides the base in the ratio of the sides containing the vertical angle.

Theorem 6.

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Case I. *If* $\angle A$ *is acute.*

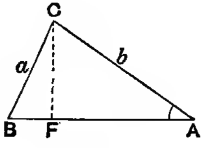

fig. **11.**

$$a^2 = b^2 + c^2 - 2c \,.\, AF. \qquad \text{II. 9.}$$

But $AF = b \cos A$,

$$\therefore\ a^2 = b^2 + c^2 - 2bc \cos A.$$

CASE II. *If* $\angle$ A *is obtuse.*

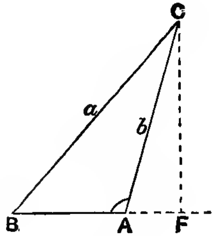

fig. 12.

$$a^2 = b^2 + c^2 + 2c \,.\, \mathrm{AF}. \qquad \text{II. 8.}$$

$$\text{But } \mathrm{AF} = b \cos \mathrm{CAF}$$

$$\text{and } \cos \mathrm{A} = -\cos \mathrm{CAF},$$

$$\therefore \mathrm{AF} = -b \cos \mathrm{A},$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos \mathrm{A}.$$

$$\text{Similarly } b^2 = c^2 + a^2 - 2ca \cos \mathrm{B},$$

$$c^2 = a^2 + b^2 - 2ab \cos \mathrm{C}.$$

EX. **23** Examine the case $\angle \mathrm{A} = 90°$.

THEOREM 7.

APOLLONIUS'* THEOREM.

If α is mid-point of BC, then

$$\mathrm{AB}^2 + \mathrm{AC}^2 = 2\mathrm{A}\alpha^2 + 2\mathrm{B}\alpha^2.$$

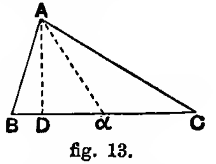

fig. 13.

Draw AD $\perp$ to BC.

Suppose that, of the $\angle$ s AαB, AαC, $\angle$ AαB is acute.

* *Apollonius* (260—200 B.C.) studied and probably lectured at Alexandria. Nicknamed ϵ.

Then, from $\triangle AB\alpha$

$$AB^2 = A\alpha^2 + B\alpha^2 - 2B\alpha \,.\, D\alpha,$$

and from $\triangle AC\alpha$

$$AC^2 = A\alpha^2 + C\alpha^2 + 2C\alpha \,.\, D\alpha.$$

But $$C\alpha = B\alpha,$$

$$\therefore\ AB^2 + AC^2 = 2A\alpha^2 + 2B\alpha^2.$$

Ex. **24.** Examine what this theorem becomes in the following cases, giving a proof in each case:

(i) if A coincides with a point in BC.

(ii) if A coincides with C.

(iii) if A coincides with a point in BC produced.

Ex. **25.** The base BC of an isosceles $\triangle ABC$ is produced to D, so that $CD = BC$; prove that $AD^2 = AC^2 + 2BC^2$.

Ex. **26.** A side PR of an isosceles $\triangle PQR$ is produced to S so that $RS = PR$: prove that $QS^2 = 2QR^2 + PR^2$.

Ex. **27.** The base AD of a triangle OAD is trisected in B, C. Prove that $OA^2 + 2OD^2 = 3OC^2 + 6CD^2$.

Ex. **28.** In the figure of Ex. 27, $OA^2 + OD^2 = OB^2 + OC^2 + 4BC^2$.

Ex. **29.** If Q is a point on BC such that $BQ = n \,.\, QC$, then

$$AB^2 + n \,.\, AC^2 = BQ^2 + n \,.\, CQ^2 + (n+1)\, AQ^2.$$

(This is a generalized theorem, of which Apollonius' theorem is a particular case. Also compare Ex. 27.)

Ex. **30.** A point moves so that the sum of the sqnares of its distances from two fixed points A, B remains constant; prove that its locus is a circle.

Ex. **31.** The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

Ex. **32.** In any quadrilateral the sum of ths squares on the fonr sides exceeds the sum of the squares on the diagonals by four times the square on the straight line joining the mid-points of the diagonals.

Ex. **33.** The sum of the squares on the diagonals of a quadrilateral is equal to twice the sum of the squares on the lines joining the mid-points of opposite sides.

Ex. **34.** In a triangle, three times the sum of the squares on the sides = four times the sum of the squares on the medians.

Definition. A set of lines which all pass through the same point are called **concurrent.**

Definition. A set of points which all lie on the same line are called **collinear.**

Definition. The circumscribing circle of a triangle is often called the **circum-circle**; and its centre the **circum-centre.**

Theorem 8.

The perpendicular bisectors of the sides of a triangle are concurrent; and the point of concurrence, S, is the circumcentre.

Every point on the $\perp$ bisector of CA is equidistant from C and A, and every point on the $\perp$ bisector of AB is equidistant from A and B.

$\therefore$ the point where these lines meet is equidistant from A, B, and C; and, being equidistant from B and C, it is on the $\perp$ bisector of BC.

$\therefore$ the $\perp$ bisectors of the three sides meet at S, the circumcentre.

Ex. **35.** Through A, B, C draw lines parallel to BC, CA, AB respectively, forming a triangle A′B′C′. By considering the circumcentre of $\triangle$A′B′C′, prove that the altitudes of $\triangle$ABC are concurrent.

Ex. **36.** Through each vertex of a triangle a pair of lines is drawn parallel to the lines joining the circumcentre to the other two vertices. Show that these six lines form an equilateral hexagon, whose opposite angles are equal.

Definition. The inscribed circle of a triangle is often called the **in-circle**; and its centre the **in-centre.**

THEOREM 9.

The internal bisectors of the angles of a triangle are concurrent; and the point of concurrence, I, is the in-centre.

Every point on the internal bisector of $\angle$ B is equidistant from AB and BC, and every point on the internal bisector of $\angle$ C is equidistant from BC and CA.

$\therefore$ the point where these lines meet is equidistant from BC, CA and AB; and, being equidistant from CA and AB and inside the triangle, it is on the internal bisector of $\angle$ A.

$\therefore$ the internal bisectors of the three angles meet at I, the in-centre.

Ex. **37.** Prove that $r=\frac{\Delta}{s}$.

[Use $\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB$.]

Ex. **38.** If a polygon is such that a circle can be inscribed in it, the bisectors of the angles are concurrent.

State a corresponding theorem for a polygon *about which* a circle can be described.

Ex. **39.** Describe a circle to touch a given circle and two of its tangents.

Ex. **40.** Prove that any circle whose centre is I cuts off equal chords from the three sides.

Ex. **41.** If AI meets the in-circle in P, prove that P is the in-centre of $\triangle$AYZ. (For notation see p. 16.)

Ex. **42.** The internal and external bisectors of $\angle$A meet the circumcircle in K, K′. Prove that KK′ is the perpendicular bisector of BC.

Ex. **43.** If AI meets the circumcircle in U, SU is perpendicular to BC.

Definition. A circle which touches one side of a triangle, and the other two sides produced, is called an **escribed circle** or an **ex-circle**. Its centre is called an **ex-centre**.

A triangle clearly has 3 ex-circles.

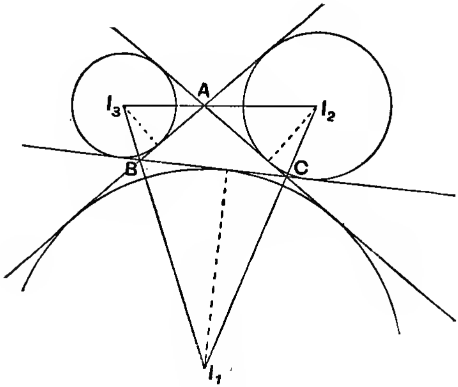

fig. 14.

THEOREM 10.

The internal bisector of $\angle$ A, and the external bisectors of $\angle$s B and C are concurrent; and the point of concurrence is the ex-centre I_1.

The proof is left to the reader.

Ex. **44.** A, I, I_1 are collinear.

Ex. **45.** I_2, A, I_3 are collinear.

Ex. **46.** AI_1 is $\perp$ to I_2I_3.

Ex. **47.** If another interior common tangent be drawn to the circles I, I_1, and cut BC in K, then IKI_1 is a straight line.

Theorem 11.

$$r = \frac{\Delta}{s}.$$

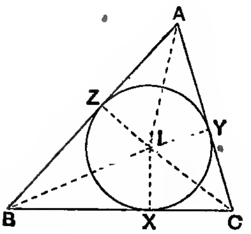

fig. 15.

$$\triangle \mathrm{IBC} + \triangle \mathrm{ICA} + \triangle \mathrm{IAB} = \triangle \mathrm{ABC}.$$

Now $\triangle \mathrm{IBC} = \frac{1}{2}\mathrm{IX}\,.\,\mathrm{BC}$

$$= \tfrac{1}{2}ra,$$

$$\triangle \mathrm{ICA} = \tfrac{1}{2}rb,$$

$$\triangle \mathrm{IAB} = \tfrac{1}{2}rc;$$

$$\therefore \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = \Delta;$$

$$\therefore r\frac{a+b+c}{2} = \Delta;$$

$$\therefore rs = \Delta;$$

$$\therefore r = \frac{\Delta}{s}.$$

Theorem 12.

$$r_1 = \frac{\Delta}{s-a}.$$

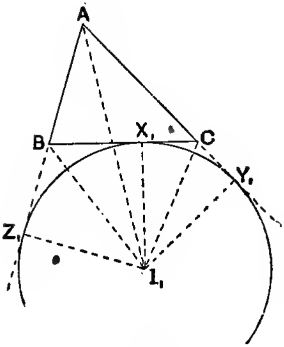

fig. 16.

$$\triangle I_1CA + \triangle I_1AB - \triangle I_1BC = \triangle ABC.$$

Now $\triangle I_1CA = \frac{1}{2}I_1Y_1 . CA$

$$= \tfrac{1}{2}r_1 b,$$

$$\triangle I_1AB = \tfrac{1}{2}r_1 c,$$

$$\triangle I_1BC = \tfrac{1}{2}r_1 a;$$

$$\therefore\ \tfrac{1}{2}r_1 b + \tfrac{1}{2}r_1 c - \tfrac{1}{2}r_1 a = \Delta;$$

$$\therefore\ r_1 \frac{b+c-a}{2} = \Delta.$$

But $b + c - a = (a + b + c) - 2a$

$$= 2s - 2a;$$

$$\therefore\ r_1(s-a) = \Delta;$$

$$\therefore\ r_1 = \frac{\Delta}{s-a}.$$

Similarly $r_2 = \dfrac{\Delta}{s-b}, \qquad r_3 = \dfrac{\Delta}{s-c}.$

Ex. **48.** Prove that in an equilateral triangle $r=\frac{1}{2}R$, $r_1=r_2=r_3=\frac{3R}{2}$.

Ex. **49.** If the ex-centres be joined, the triangle so formed is similar to the triangle XYZ.

Ex. **50.** Prove that the circle on II_1 as diameter passes through B and C. Hence construct a triangle, having given BC, $\angle$ B, and the length II_1.

Ex. **51.** $AZ_1+AY_1=AB+AC+BC$.

Ex. **52.** $AY_1=AZ_1=s$.

Ex. **53.** $AZ+AY=AB+AC-BC$.

Ex. **54.** $AY=AZ=s-a$.

Ex. **55.** $ZZ_1=YY_1=a$.

Ex. **56.** $BX_1=CX=s-c$.

Ex. **57.** $BX=CX_1=s-b$.

Ex. **58.** $XX_1=c \sim b$.

Theorem 13.

(i) $AY_1=AZ_1=s$. (ii) $AY=AZ=s-a$.

(iii) $YY_1=ZZ_1=a$. (iv) $BX_1=CX=s-c$.

(v) $XX_1=c \sim b$.

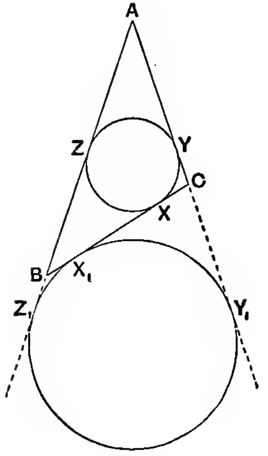

fig. 17.

(i) $$\begin{aligned} AY_1 + AZ_1 &= AC + CY_1 + AB + BZ_1 \\ &= AC + CX_1 + AB + BX_1 \text{ (since tangents to a circle from a point are equal)} \\ &= AC + AB + BC \\ &= 2s. \end{aligned}$$

But $AY_1 = AZ_1$,

$\therefore$ $AY_1 = AZ_1 = s$.

(ii) $$\begin{aligned} AY + AZ &= AC - CY + AB - BZ \\ &= AC - CX + AB - BX \\ &= AC + AB - BC \\ &= 2s - 2a. \end{aligned}$$

But $AY = AZ$,

$\therefore$ $AY = AZ = s - a$.

(iii) $$\begin{aligned} YY_1 &= AY_1 - AY \\ &= s - (s - a) \\ &= a. \end{aligned}$$

Similarly $ZZ_1 = a$.

(iv) $$BX_1 = BZ_1 = AZ_1 - AB$$
$$= s - c.$$

Also $CX = s - c$, by proof similar to (ii).

(v) $$XX_1 = BC - CX - BX_1$$
$$= a - 2(s - c)$$
$$= a - (a + b + c) + 2c$$
$$= c - b.$$

If the figure were drawn with $b > c$, it would be found that $XX_1 = b - c$.

Ex. 59. Find the lengths of the segments into which the point of contact of the in-circle divides the hypotenuse of a right-angled triangle whose sides are 6 and 8 feet.

Ex. 60. The distance between X and the mid-point of BC is $\frac{1}{2}(b \sim c)$.

Ex. 61. The in-radius of a right-angled triangle is equal to half the difference between the sum of the sides and the hypotenuse.

Ex. 62. If the diagonals of a quadrilateral ABCD intersect at right angles at O, the sum of the in-radii of the triangles AOB, BOC, COD, DOA is equal to the difference between the sum of the diagonals and the semi-perimeter of the quadrilateral. (Use Ex. 61.)

Ex. 63. Two sides of a triangle of constant perimeter lie along two fixed lines; prove that the third side touches a fixed circle.

Definition. The line joining a vertex of a triangle to the mid-point of the opposite side is called a **median.**

Definition. The triangle whose vertices are the mid-points of the sides is called the **medial triangle** of the given triangle.

Ex. 64. Prove that two medians trisect one another.

Ex. 65. Hence prove that the three medians are concurrent.

Ex. 66. The circumradius of the medial triangle is $\frac{1}{2}$R.

Lemma 1.

If γ, β are the mid-points of AB and AC, then $\gamma\beta$ is parallel to BC and equal to $\frac{1}{2}$BC.

The proof is left to the reader.

Theorem 14.

The medians of a triangle are concurrent; and each median is trisected at the point of concurrence, G.

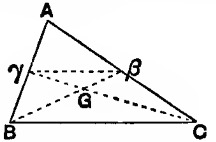

fig. 18.

Let the two medians $B\beta$, $C\gamma$ meet at G.

Join $\beta\gamma$.

Then, by Lemma 1, $\gamma\beta$ is $\parallel$ to BC and $=\frac{1}{2}$BC.

Again, $\triangle$s $G\beta\gamma$, GBC are similar (why?),

$$\therefore\ G\beta : GB = G\gamma : GC$$
$$= \beta\gamma : BC = 1 : 2.$$

$\therefore$ two medians $B\beta$, $C\gamma$ intersect at a point of trisection of each.

Let the median $A\alpha$ cut $B\beta$ in G'.

Then it may be proved, as above, that $\beta G' = \frac{1}{3}\beta B$, $\alpha G' = \frac{1}{3}\alpha A$.

$$\text{But } \beta G = \tfrac{1}{3}\beta B,$$
$$\therefore\ G' \text{ coincides with } G,$$
$$\text{and } \alpha G = \tfrac{1}{3}\alpha A.$$

$\therefore$ the three medians are concurrent and each median is trisected at the point of concurrence, G.

Note. It will be noticed that G, the point of concurrence of the three medians, is the **centroid** of the three points A, B, C (see Chap. III.): accordingly G is called the **centroid of the triangle.**

Ex. **67.** Prove the centroid property of a triangle by the following method: let $B\beta$, $C\gamma$ meet in G; produce AG to P so that GP = AG: then prove that GBPC is a $\parallel^{\text{ogram}}$ etc.

Ex. **68.** The triangles GBC, GCA, GAB are equivalent.

Ex. **69.** On AB, AC points Q, R are taken so that $AQ=\frac{1}{3}AB$, $AR=\frac{1}{3}AC$. CQ, BR meet in P, and AP produced meets BC in D; find the ratio AP : AD.

Ex. **70.** The triangles ABC, $\alpha\beta\gamma$ have the same centroid.

THEOREM 15.

The three altitudes of a triangle are concurrent.

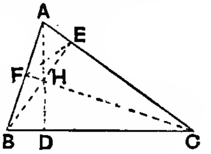

fig. 19.

Draw BE, CF $\perp$ to AC, AB; let them meet in H. Join AH and produce it to meet BC in D.

We have to prove that AD is $\perp$ to BC.

Join FE.

Since $\angle$s AFH, AEH are rt. $\angle$s,

A, F, H, E are concyclic;

$\therefore$ $\angle$ FAH = $\angle$ FEH.

Again, since $\angle$s BFC, BEC are rt. $\angle$s,

$\therefore$ B, F, E, C are concyclic;

$\therefore$ $\angle$ FEH = $\angle$ FCB.

But $\angle$ FAH = $\angle$ FEH,

$\therefore$ $\angle$ FAH = $\angle$ FCB,

$\therefore$ F, A, C, D are concyclic,

$\therefore$ $\angle$ ADC = $\angle$ AFC = a rt. $\angle$.

Hence AD is $\perp$ to BC,

and the three altitudes are concurrent.

Ex. **71.** Does the above proof need any modification if $\angle$ A is right or obtuse?

Definition. The point of concurrence, H, of the altitudes of a triangle is called the **orthocentre.**

Ex. **72.** If H is the orthocentre of $\triangle$ ABC, then A is the orthocentre of $\triangle$ BCH, B of $\triangle$ CAH, and C of $\triangle$ ABH.

Ex. **73.** I is the orthocentre of $\triangle I_1I_2I_3$.

(Notice that A, I, I_1 are collinear; as also I_2, A, I_3.)

Ex. **74.** AH . HD = BH . HE = CH . HF.

Ex. **75.** AS and AH are equally inclined to the bisector of $\angle$ A.

Ex. **76.** $\angle$ BHC is the supplement of $\angle$ A.

Ex. **77.** Show that if two of the opposite angles of a convex quadrilateral be right angles, the external diagonal of the complete quadrilateral formed by the sides is perpendicular to an internal diagonal.

Definition. The triangle whose vertices are the feet of the altitudes is called the **pedal triangle** of the given triangle.

Ex. **78.** The triangles ABC, HBC, HCA, HAB all have the same pedal triangle.

Ex. **79.** The orthocentre of a triangle is the in-centre or an ex-centre of its pedal triangle.

Ex. **80.** The triangle formed by the tangents at A, B, C to the circumcircle is similar and similarly situated to the pedal triangle.

THEOREM 16.

If AH produced meets the circumcircle in X, then HD = DX.

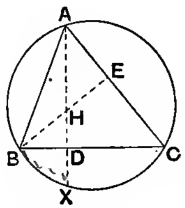

fig. 20.

Since $\angle$s E and D are rt. $\angle$s,

$\therefore$ A, E, D, B are concyclic,

$\therefore$ $\angle$ DBE = $\angle$ DAE.

Also $\angle$ DBX = $\angle$ DAE, in the same segment.

$\therefore$ $\angle$ DBE = $\angle$ DBX.

Hence $\triangle$s DBH, DBX are congruent,

and HD = DX.

Ex. 81. Draw a figure for Theorem 16, in which $\angle$ A is obtuse. Does the proof need any modification for this case?

Ex. 82. The triangles ABC, AHB, BHC, CHA have equal circum-circles.

Ex. 83. H is the circumcentre of the triangle formed by the circum-centres of AHB, BHC, CHA.

Ex. 84. BD . DC = AD . HD.

Ex. 85. The base and vertical angle of a triangle are given. Prove that the locus of the orthocentre is a circle equal to the circumcircle. Find also the looi of the in-centre and the centroid.

Theorem 17.

AH = 2Sα.

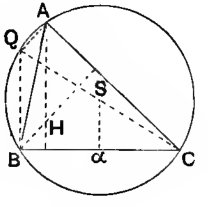

fig. 21.

Let CS meet circumcircle in Q.

Since S and α are the mid-points of CQ and CB respectively,

QB = 2Sα,

and QB is $\parallel$ to Sα and to AH.

Again, as CQ is a diameter, $\angle$ CAQ is a rt. $\angle$,

$\therefore$ AQ is $\parallel$ to HB.

Hence AQBH is a $\parallel^{\text{ogram}}$.

$\therefore$ AH = QB = 2Sα.

Ex. **86.** Prove Theorem 17 by using the fact that H is the circumcentre of the triangle formed by drawing parallels to the sides through the opposite vertices.

Ex. **87.** Let P be the mid-point of AH. Show that αP, SH bisect one another.

Ex. **88.** Show that N, the mid-point of HS, is the centre of the circle PDα.

Ex. **89.** Show that αP is equal to the circumradius of ABC.

Ex. **90.** Show that a circle with centre N (the mid-point of HS) and radius equal to $\frac{1}{2}$R will pass through D, E, F, α, β, γ and the mid-points of HA, HB, HC.

Ex. **91.** The perpendicular bisectors of Dα, Eβ, Fγ are concurrent.

Ex. **92.** Prove that AS, Hα meet on the circumcircle.

Ex. **93.** If P, Q, R are the mid-points of HA, HB, HC, then $\triangle$PQR is congruent with $\triangle\alpha\beta\gamma$.

Ex. **94.** SP is bisected by the median Aα.

Ex. **95.** The circumradius of $\triangle\alpha\beta\gamma$ is $\frac{1}{2}$R.

Ex. **96.** Prove that $\triangle$s $\alpha\beta\gamma$, D$\gamma\beta$ are congruent.

Ex. **97.** Show that $\alpha\beta\gamma$D are concyclic. Use Ex. 96 to show that the circumcircle of the pedal triangle passes through the mid-points of the sides.

Ex. **98.** Apply the result of Ex. 97 to the triangle HBC.

Ex. **99.** Combining the two preceding exercises, deduce the result of Ex. 90.

Theorem 18.

The points H, G, S are collinear; and HG = 2GS.

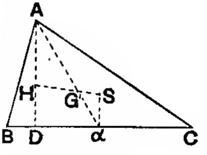

fig. 22.

Let $A\alpha$ cut HS in G'.

Since AH and $S\alpha$ are $\|$, $\triangle$s AHG', $\alpha SG'$ are similar,

and since $AH = 2S\alpha$,

$\therefore\ AG' = 2G'\alpha$.

$\therefore\ G'$ is identical with G, the centroid.

Also $HG = 2GS$.

Ex. **100.** Use fig. 22 to prove, independently, the concurrence of the three medians.

Ex. **101.** If AS, $H\alpha$ meet at K, the centroid of $\triangle$AKH is G.

Theorem 19.

A circle whose centre is the mid-point of SH, and whose radius is $\frac{1}{2}$R, passes through

D, E, F the feet of the altitudes,

α, β, γ the mid-points of the sides,

P, Q, R the mid-points of HA, HB, HC.

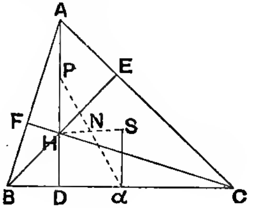

fig. 23.

Join αP, SH. Let them intersect at N.

(i) $HP = \frac{1}{2}HA = \alpha S$, and HP is $\|$ to αS,

$\therefore$ $HPS\alpha$ is a $\|^{\text{ogram}}$,

and the diagonals HS, $P\alpha$ bisect one another.

$\therefore$ N is the mid-point of HS and bisects $P\alpha$.

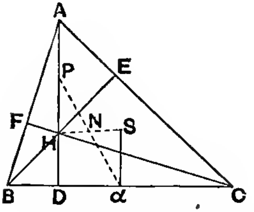

fig. 23.

(ii) Since $\angle$ PDα is a rt. $\angle$, Pα is the diameter and N the centre of $\odot$ PDα.

(iii) AP is equal and $\parallel$ to Sα,

$\therefore$ APαS is a $\parallel^{\text{ogram}}$,

$\therefore$ αP = SA,

and NP, the radius of $\odot$ PD$\alpha = \frac{1}{2}\alpha$P $= \frac{1}{2}$SA $= \frac{1}{2}$R.

(iv) It has been shown that the circle whose centre is N, the mid-point of SH, and whose radius is $\frac{1}{2}$R, passes through the foot of *one* altitude, the mid-point of *one* side, and the mid-point of HA.

By similar reasoning this circle may be shown to pass through the feet of the three altitudes, the mid-points of the three sides, and the mid-points of HA, HB, HC.

This circle is called the **nine-points circle**, and its centre N is called the **nine-points centre**.

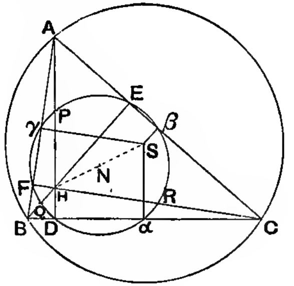

fig. 24.

Ex. **102.** The circumcircle of $\triangle$ ABC is the 9-points circle of $\triangle I_1 I_2 I_3$.

Ex. **103.** The circumcircle bisects each of the 6 lines joining pairs of the points I, I_1, I_2, I_3.

Ex. **104.** If O be equidistant from I_1, I_2, I_3, then S is the mid-point of OI.

Ex. **105.** What is the 9-points circle of $\triangle$ BHC?

Ex. **106.** P is any point on the circumcircle of $\triangle$ABC. PL, PM, PN are $\perp$ to BC, CA, AB respectively. Prove that

(i) $\angle$ PNL $= 180° - \angle$ PBC.
(ii) $\angle$ PNM $= \angle$ PAM.
(iii) $\angle$ PNL $+ \angle$ PNM $= 180°$.
(iv) LNM is a straight line.

Theorem 20.

If from P, a point on the circumcircle, perpendiculars PL, PM, PN be drawn to the sides of a triangle, then LMN is a straight line (the Simson* line).

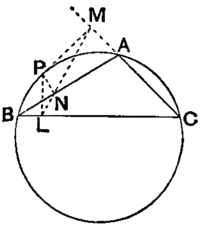

fig. 25.

Join LN, NM.

Since $\angle$s PNB, PLB are rt. $\angle$s,

$\therefore$ $\angle$ PNL $= 180° - \angle$ PBC.

Again, since $\angle$s PNA, PMA are rt. $\angle$s,

$\therefore$ $\angle$ PNM $= \angle$ PAM.

But $\angle$ PAM $= 180° - \angle$ PAC

$= \angle$ PBC,

$\therefore$ $\angle$s PNL, PNM are supplementary,

$\therefore$ LNM is a straight line.

* Robert Simson (1687–1768), professor of mathematics at Glasgow; author of several works on ancient geometry, and, in particular, of an edition of Euclid's *Elements* on which most modern editions are based.

Ex. **107.** State and prove a true converse of Th. 20.

Ex. **108.** Draw a figure for Th. 20 with P on arc BC; does the proof need any modification?

Ex. **109.** What is the Simson line of A? of the point on the circumcircle diametrically opposite to A?

Ex. **110.** AD meets the circumcircle in X; the Simson line of X is parallel to the tangent at A.

Ex. **111.** AI meets the circumcircle in U; the Simson line of U bisects BC.

Ex. **112.** If PL meets the circumcircle in U, AU is parallel to the Simson line.

Ex. **113.** The altitude from A is produced to meet the circumcircle in X, and X is joined to a point P on the circumcircle. PX meets the Simson line of P in R; and BC in Q. Prove that R is the mid-point of PQ.

Ex. **114.** In Ex. 113 show that HQ is parallel to the Simson line of P.

Ex. **115.** **From Ex. 114 deduce that the line joining a point on the circumcircle to the orthocentre is bisected by the Simson line of the point.**

Ex. **116.** Prove the last exercise with the following construction: take image p of P in BC; join pH, PX, and prove pH parallel to the Simson line of P.

Ex. **117.** Given four straight lines, find a point such that its projections on the four lines are collinear.

Ex. **118.** Given four straight lines, prove that the circumcircle of the four triangles formed by the lines have a common point. Show that this is the focus of the parabola that touches the four lines.

Exercises on Chapter IV.

Ex. **119.** Given the base, the circumradius, and the difference of the base angles of a triangle, show how to construct the triangle.

Ex. **120.** Two vertices B, C of a triangle are fixed, and the third vertex A moves in a straight line through B. Prove that the locus of the orthocentre is a straight line. What is the locus of the circumcentre? of the incentre? of the centroid? of the point where the altitude from A meets the circumcircle?

Ex. **121.** If a series of trapezia be formed by drawing parallels to the base of a triangle, the locus of the intersections of the diagonals of these trapezia is a median of the triangle.

Ex. **122.** The base BC of a triangle ABC is divided at P, so that $m\mathrm{BP}=n\mathrm{PC}$; prove that

$$m\mathrm{AB}^2+n\,\mathrm{AC}^2=(m+n)\,(\mathrm{AP}^2+\mathrm{BP}\,.\,\mathrm{PC}).$$

Ex. **123.** The lines joining the circumcentre to the vertices of a triangle are perpendicular to the sides of the pedal triangle.

Ex. **124.** Construct a triangle, given:

(i) two sides and a median (2 cases),

(ii) a side and two medians (2 cases),

(iii) the three medians,

(iv) the base, the difference of the two sides, and the difference of the base angles,

(v) the base, a base angle, and the sum or difference of the two other sides,

(vi) the base, the vertical angle, and the sum or difference of the two other sides,

(vii) the feet of the three perpendiculars,

(viii) an angle, an altitude and the perimeter (2 cases),

(ix) a side, one of the adjacent angles, and the length of the bisector of this angle,

(x) the sum of two sides, and the angles,

(xi) the perimeter and the angles,

(xii) an angle, the length of its bisector, and one of the altitudes (2 cases),

(xiii) the angles and an altitude,

(xiv) the base, the sum of two other sides, and the difference of the base angles.

Ex. **125.** Construct a triangle having given the orthocentre, the circumcentre, and the position (not length) of one of the sides.

Ex. **126.** Construct a triangle given the circumcircle, the orthocentre and one vertex.

Ex. **127.** The magnitude of the angle A of a triangle ABC, and the lengths of the two medians which pass through A and B are known. Construct the triangle.

Ex. **128.** The median through A of the triangle AEF makes the same angle with AB as does Aα with AC.

Ex. **129.** If perpendiculars OX, OY, OZ be drawn from any point O to the sides BC, CA, AB of a triangle,

$$BX^2+CY^2+AZ^2=CX^2+AY^2+BZ^2.$$

State and prove a converse theorem.

Ex. **130.** If I_1X, I_2Y, I_3Z be drawn perpendicular to BC, CA, AB respectively, these three lines are concurrent.

Ex. **131.** Let AI produced meet the circumcircle in K. Prove that

$$KB=KC=KI.$$

Draw KK′, a diameter of the circumcircle; and draw IY ⊥ to AC. Prove that △s K′KC, AIY are similar.

Hence show that IA . IK$=2Rr$; i.e. that the rectangle contained by the segments of any chord of the circumcircle drawn through the incentre$=2Rr$.

Ex. **132. From Ex. 131 deduce that $SI^2=R^2-2Rr$.**

Ex. **133.** Upon a given straight line AB any triangle ABC is described having a given vertical angle ACB. AD, BE are the perpendiculars from A, B upon the sides BC, CA meeting them in D and E respectively. Prove that the circumcentre of the triangle CED is at a constant distance from DE.

Ex. **134.** The triangle ABC has a right angle at C, and AEFB, ACGH are squares described outside the triangle. Show that if K be taken on AC (produced if necessary) so that AK is equal to BC, then A is the centroid of the triangle HEK.

Ex. **135.** If four circles be drawn, each one touching three sides of a given quadrilateral, the centres of the four circles are concyclic.

Ex. **136.** The orthocentre of a triangle ABC is H, and the midde point of BC is D. Show that DH meets the circumcircle at the end of the diameter through A, and also at the point of intersection of the circumcircle with the circle on AD as diameter.

Ex. **137.** ABC is a triangle, right-angled at A; DEF is a straight line perpendicular to BC, and cutting BC, CA, AB in E, F, D respectively. BF, CD meet at P. Find the locus of P.

Ex. **138.** Two fixed tangents OP, OQ are drawn to a fixed circle; a variable tangent meets the fixed tangents in X, Y. Prove (i) that the perimeter of the triangle OXY is constant, (ii) that XY subtends a fixed angle at the centre of the circle.

Ex. **139.** Prove that $\angle$SAH is the difference between the angles B and C. Hence construct a triangle, having given the nine-points circle, the orthocentre, and the difference between two of its angles. Is there any ambiguity?

Ex. **140.** The lines joining I to the ex-centres are bisected by the circumcircle.

Ex. **141.** The circle BIC cuts AB, AC in E, F; prove that EF touches the in-circle.

Ex. **142.** The triangle formed by the circumcentres of AHB, BHC, CHA is congruent with ABC.

Ex. **143.** Through C, the middle point of the arc ACB of a circle, any chord CP is drawn, cutting the straight line AB in Q. Show that the locus of the centre of the circle circumscribing the triangle BQP is a straight line.

Ex. **144.** A circle is escribed to the side BC of a triangle ABC touching the other sides in F and G. A tangent DE is drawn parallel to BC, meeting the sides in D, E. DE is found to be three timee BC in length. Show that DE is twice AF.

Ex. **145.** Two triangles ABC, DEF are inscribed in the same circle so that AD, BE, CF meet in a point O; prove that, if O be the in-centre of one of the triangles, it will be the orthocentre of the other.

Ex. **146.** If equilateral triangles be described on the sides of a triangle (all outside or all inside), the lines joining the vertices of the triangle to the vertices of the opposite equilateral triangles are equal and concurrent.

Ex. **147.** If on the sides of any triangle three equilateral triangles be constructed, the in-centres of these triangles form another equilateral triangle.

Ex. **148.** Construct a triangle having given the centres of its inscribed circle and of two of its ex-circles.

Ex. **149.** The circumcentre of the triangle BI_1C lies on the circumcircle of ABC.

Ex. **150.** Construct a triangle given the base, vertical angle and in-radius.

Ex. **151.** A pair of common tangents to the nine-points circle and circumcircle meet at the orthocentre.

Ex. **152.** On the sides AB, AC of a triangle ABC any two points N, M are taken concyclic with B, C. If NC, MB intersect in P, then the bisector of the angle between AP and the line joining the middle points of BC, AP makes a constant angle with BC.

Ex. **153.** Any line from the orthocentre to the circumference of the circumcircle is bisected by the nine-points circle.

Ex. **154.** If P be any point on the circumcircle and parallels to PA, PB, PC respectively be drawn through α, β, γ, the mid-points of the sides, prove that these parallels intersect in the same point on the nine-points circle.

Ex. **155.** If perpendiculars are drawn from the orthocentre of a triangle ABC on the bisectors of the angle A, show that their feet are collinear with the middle point of BC.

Ex. **156.** If two circles are such that one triangle can be inscribed in the one and circumscribed to the other, show that an infinite number of such triangles can be so constructed.

Prove that the locus of the orthocentre of these triangles is a circle.

Ex. **157.** The triangle ABC has a right angle at A. AD is the perpendicular from A on BC. O, O′ are the centres of the circles inscribed in the triangles ABD, ACD respectively. Prove that the triangle ODO′ is similar to ABC.

Ex. **158.** If D, E, F be the feet of the perpendiculars from a point on the circumcircle upon the sides, find the position of the point so that DE may be equal to EF.

Ex. **159.** From P, a point on the circumcircle of a triangle ABC, perpendiculars PL, PM, PN are drawn to the sides. Prove that the rectangles PL . MN, PM . NL, PN . LM are proportional to the sides BC, CA, AB.

Ex. **160.** The Simson line of a point P rotates at half the rate at which P rotates about the centre of the circle.

Ex. **161.** The Simson lines of opposite ends of a diameter of the circumcircle are at right angles to one another.

Ex. **162.** Find the three points on the circle circumscribing the triangle ABC such that the pedal lines of the points with respect to the triangle are perpendicular to the medians of the triangle.

Ex. **163.** P, Q, R are three points taken on the sides BC, CA, AB respectively of a triangle ABC. Show that the circles circumscribing the triangles AQR, BRP, CPQ meet at a point, which is fixed relatively to the triangle ABC if the angles of the triangle PQR are given.

If PQR is similar to ABC show that this point is the orthocentre of PQR and the circumcentre of ABC.

Ex. **164.** A straight line AB of constant length has its extremities on two fixed straight lines OX, OY respectively. Show that the locus of the orthocentre of the triangle OAB is a circle.

Ex. **165.** Find the locus of a point such that its projections upon three given straight lines are collinear.

Ex. **166.** The circumcircle of the triangle formed by any three of the four common tangents to two circles passes through the middle point of the line joining their centres.

Ex. **167.** If one of the angles of the triangle be half a right angle, prove that the line joining the orthocentre to the centre of the circumcircle is bisected by the line joining two of the feet of the perpendiculars from the angles upon the opposite sides.

Ex. **168.** B, C are fixed points, A a variable point on a fixed circle which passes through B and C. Show that the centres of the four circles which touch the sides of the triangle ABC are at the extremities of diameters of two other fixed circles.

Ex. **169.** The bisector of the angle BAC meets BC in Y; X is the point on BC such that BX=YC, XC=BY; prove that

$$AX^2-AY^2=(AB-AC)^2.$$

Ex. **170.** A straight line PQ is drawn parallel to AB to meet the circumcircle of the triangle ABC in the points P and Q; show that the pedal lines of P and Q intersect on the perpendicular from C on AB.

Ex. **171.** From a point P on the circumcircle of a triangle are drawn lines meeting the sides in L, M, N, and making with the perpendiculars to these sides equal angles in the same sense. Show that L, M, N are collinear. What does this theorem lead to when the equal angles are 90°?

Ex. **172.** If, with a given point P, lines LMN, L′M′N′ are drawn as in the preceding exercise, by taking angles θ, θ', prove that the angle between LMN and L′M′N′ is $\theta-\theta'$.

Ex. **173.** Prove that the envelope of all lines LMN (see Ex. 171) obtained from a fixed point P by varying the angle is a parabola with focus P and touching the sides of the triangle. What relation does the Simson line bear to this parabola?

Ex. **174.** Prove that all triangles inscribed in the same circle equiangular to each other are equal in all respects.

Ex. **175.** The altitude of an equilateral triangle is equal to a side of an equilateral triangle inscribed in a circle described on one of the sides of the original triangle as diameter.

Ex. **176.** ABC, A′B′C′ are two triangles equiangular to each other inscribed in a circle AA′BB′CC′. The pairs of sides BC, B′C′; CA, C′A′; AB, A′B′ intersect in a, b, c respectively.

Prove that the triangle abc is equiangular to the triangle ABC.

Ex. **177.** Prove that all triangles described about the same circle equiangular to each other are equal in all respects.

Ex. **178.** If ABC, A′B′C′ be two equal triangles described about a circle in the same sense; and the pairs of sides BC, B′C′; CA, C′A′; AB, A′B′ meet in *a*, *b*, c respectively; then *a*, *b*, *c* are equidistant from the centre of the circte.

Ex. **179.** P is a point on the circle circumscribing the triangle ABC. The pedal line of P cuts AC and BC in M and L. Y is the foot of the perpendicular from P on the pedal line. Prove that the rectangles PY, PC, and PL, PM are equal.

CHAPTER V.

THE THEOREMS OF CEVA AND MENELAUS.

LEMMA 2.

If two triangles have the same height, their areas are to one another in the ratio of their bases.

The proof is left to the reader.

THEOREM 21.

(THE THEOREM OF CEVA*.)

If the lines joining a point O to the vertices of a triangle ABC meet the opposite sides in X, Y, Z, then $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1$, the sense of lines being taken into account.

* The theorem was first published by Giovanni Ceva, an Italian, in 1678.

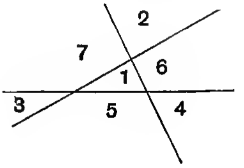

fig. 26.

By drawing various figures and placing the point O in the 7 possible different regions, the reader may see that of the ratios $\frac{BX}{CX}$, $\frac{CY}{AY}$, $\frac{AZ}{BZ}$, either 3 or 1 must be negative. The product therefore is negative; and, for the rest, it is sufficient to confine our attention to the numerical values of the ratios.

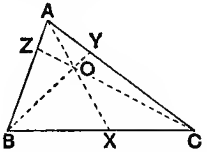

fig. 27.

$$\frac{BX}{CX} = \frac{\triangle ABX}{\triangle ACX} = \frac{\triangle OBX}{\triangle OCX} \quad \text{.........Lemma 2}$$

$$= \frac{\triangle ABX - \triangle OBX}{\triangle ACX - \triangle OCX}$$

$$= \frac{\triangle AOB}{\triangle AOC}.$$

Similarly $\frac{CY}{AY} = \frac{\triangle BOC}{\triangle BOA}$,

$$\frac{AZ}{BZ} = \frac{\triangle COA}{\triangle COB}.$$

$$\therefore \frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1 \quad \text{(numerically)},$$

$= -1$ when sense is taken into account.

THEOREM 22.

(CONVERSE OF CEVA'S THEOREM.)

If points X, Y, Z are taken on the sides BC, CA, AB of a triangle, such that $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1$, then are AX, BY, CZ concurrent.

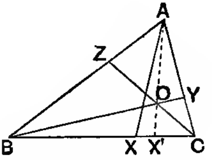

fig. 28.

If AX, BY, CZ are not concurrent, let BY, CZ meet in O, and let AO (produced if necessary) meet BC in X'.

Then $\frac{BX'}{CX'} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1$. .... (Ceva.)

But $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1$.

$\therefore \frac{BX'}{CX'} = \frac{BX}{CX}$ (sense being taken into account).

$\therefore$ X' coincides with X,

and AX, BY, CZ are concurrent.

Ex. **180.** If $\frac{BX'}{CX'} = \frac{BX}{CX}$, where sense is *not* taken into account, can it be inferred that X' coincides with X?

Ex. **181.** Using Ceva or its converse (be careful to state which you are using), prove the concurrence

(i) of the medians of a triangle;

(ii) of the bisectors of its angles;

(iii) of its altitudes.

Ex. **182.** If AZ : ZB = AY : YC, show that the line joining A to the intersection of BY and CZ is a median.

Ex. **183.** X, X′ are points on BC such that $BX = X'C$. The points Y, Y′; Z, Z′ are similarly related pairs of points on CA, AB. If AX, BY, CZ are concurrent, so also are AX′, BY′, CZ′.

Ex. **184.** The lines joining the vertices to the points of contact of the in-circle with the opposite sides are concurrent.

Ex. **185.** The lines joining the vertices to the points of contact of the corresponding ex-circles with the opposite sides are concurrent.

Theorem 23.

(The Theorem of Menelaus *.)

If a straight line cuts the sides of a triangle ABC in L, M, N, then $\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = +1$, the sense of lines being taken into account.

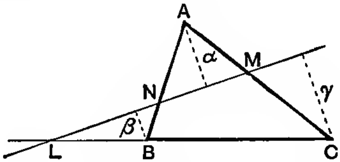

fig. 29.

As in Ceva's theorem, the reader may satisfy himself that of the ratios $\frac{BL}{CL}, \frac{CM}{AM}, \frac{AN}{BN}$, either 2 or 0 are negative. The product therefore is positive. For the rest of the proof the sense of lines will be disregarded.

Let the perpendiculars from A, B, C upon LMN be of lengths α, β, γ.

Then $$\frac{BL}{CL} = \frac{\beta}{\gamma}, \quad \frac{CM}{AM} = \frac{\gamma}{\alpha}, \quad \frac{AN}{BN} = \frac{\alpha}{\beta}.$$

$$\therefore \frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = 1 \text{ numerically}$$

$$= +1 \text{ when sense is taken into account.}$$

* *Menelaus* of Alexandria, about 98 A.D.

THEOREM 24.

(THE CONVERSE OF MENELAUS' THEOREM.)

If points L, M, N are taken on the sides BC, CA, AB of a triangle, such that $\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = +1$, then are L, M, N collinear.

The proof is left to the reader.

Ex. **186.** Prove theorem 24.

Ex. **187.** Use the above theorems to prove the theorem of the Simson line (Th. 20).

[Let $\angle PAB = \theta$, then $AN = AP\cos\theta$, etc.]

Ex. **188.** If points Q, R are taken on AB, AC so that $AQ = 2QB$, $AR = \frac{1}{2}RC$, and QR produced meets BC in P, find PB : PC.

Ex. **189.** The bisectors of $\angle$s B and C meet the opposite sides in Q, R, and QR meets BC in P; prove that AP is the exterior bisector of $\angle$ A.

Ex. **190.** α, β, γ are the mid-points of the sides; $A\alpha$ meets $\beta\gamma$ in P; CP meets AB in Q. Show that $AQ = \frac{1}{3}AB$.

EXERCISES ON CHAPTER V.

Ex. **191.** A straight line cuts the sides BC, CA, AB of a triangle in L, M, N respectively. The join of A to the intersection of BM, CN meets BC in P. Show that BC is divided in the same ratio at L and P.

Ex. **192.** The sides BC, CA, AB of a triangle ABC are divided internally by points A′, B′, C′ so that $BA' : A'C = CB' : B'A = AC' : C'B$. Also B′C′ produced cuts BC externally in A″. Prove that

$$BA'' : CA'' = CA'^2 : A'B^2.$$

Ex. **193.** Points P, P′ are taken on BC such that $PB = CP'$, and CB, AB, AC are bisected in O, K, L respectively. Prove that the intersections of OL with AP and of KP with LP′ are collinear with B.

Ex. **194.** X is any point on II_1; BX, CX meet AC, AB in Q, R; QR meets BC in U. Show that UI_2I_3 is a straight line.

Ex. **195.** The lines EF, FD, DE, which join the points of contact D, E, F of the inscribed circle of a triangle with the sides, cut the opposite sides in X, Y, Z. Prove that X, Y, Z are collinear.

Ex. **196.** A transversal through P, on BC produced, cuts off equal lengths BR, CQ from the sides AB, AC of a triangle. Show that

$$PQ : PR = AB : AC.$$

Ex. **197.** If AD, BE, CF are concurrent straight lines meeting the sides of the triangle ABC in D, E, F respectively, and the circle DEF cuts the sides again in D′, E′, F′, prove that AD′, BE′, CF′ are concurrent.

Ex. **198.** Through a point F on the diagonal BD of a square ABCD lines are drawn parallel to the sides to meet AB in G, BC in E, CD in K, and DA in H. Prove that BH, CF, and DG are concurrent.

Ex. **199.** ABC is a triangle right-angled at C; P is any point on AB. Perpendiculars are let fall from P on CA and CB. The line joining the feet of these perpendiculars meets AB in Q. Prove that $2PO . PQ = PA . PB$ where O is the mid-point of AB.

Ex. **200.** DEF is the pedal triangle of ABC; O lies on AD; OE, OF meet DF, DE in Y, Z. Show that FE, YZ, BC are concurrent.

Ex. **201.** S is a point on the side QR of a triangle PQR. The lines joining S to the mid-points of PQ, PR meet PR, PQ at T, U respectively. TU meets QR at V. Prove that $QV : RV = SQ^2 : RS^2$.

Ex. **202.** If the in-circle touch AB in Z, and the circle escribed to BC touch AC in Y_1, then ZY_1 is divided by BC in the ratio AC : AB.

Ex. **203.** A line drawn through the vertex A of a square ABCD meets the sides BC, CD in E and F; DE and BF meet in G; CG meets AD in H. Prove that DF = DH.

Ex. **204.** The sides AB, CD of a quadrilateral ACDB are parallel; CA, DB meet in E, CB, AD meet in H, and CB, AD meet FEG, a parallel to AB, in G and F respectively. Show that AG, BF, and EH are concurrent.

Ex. **205.** The line CF cuts the side AB of a triangle ABC in a point F such that AF : FB $= n : 1$; and lines are drawn through A and B parallel to the opposite sides. Show that the ratios of the area of the triangle formed by these lines and CF to the area of the triangle ABC is $(1-n)^2 : n$.

Ex. **206.** D, E, F are points on the sides of a triangle ABC, and AD, BE, CF meet in O. Prove that

$$\frac{OD}{AD}+\frac{OE}{BE}+\frac{OF}{CF}=1.$$

CHAPTER VI.

HARMONIC SECTION.

[*Throughout this chapter, the sense of lines will be taken into account.*]

Definition. If a straight line AB is divided at two points C, D so that $\frac{AC}{CB}\Big/\frac{AD}{DB}=-1$, it is said to be **divided harmonically**; A, C, B, D are said to form a **harmonic range**; and C and D are called **harmonic conjugates** with respect to A and B.

Note that the above definition is the same as the following. If a straight line is divided internally and externally in the same ratio, it is said to be divided harmonically.

Ex. **207.** Take a line AB 6 cm. long; divide it at C so that $\frac{AC}{CB}=2$; find the point D such that C, D divide AB harmonically.

Ex. **208.** Repeat Ex. 207 with (i) $\frac{AC}{CB}=\frac{1}{2}$, (ii) $\frac{AC}{CB}=-2$, (iii) $\frac{AC}{CB}=-\frac{1}{2}$.

Ex. **209. If AB is divided harmonically at C, D, then CD is divided harmonically at A, B.**

Ex. **210.** Draw a scalene triangle ABO; draw the internal and external bisectors of the angle at O and let them cut the base in C and D. Calculate (from actual measurements) $\frac{AC}{CB}\Big/\frac{AD}{DB}$. Is A, C, B, D a harmonic range?

Ex. **211. Prove that the internal and external bisectors of an angle of a triangle divide the opposite side of the triangle harmonically.**

Definition. If A, C, B, D be any four points in a straight line, $\dfrac{AC}{CB}\Big/\dfrac{AD}{DB}$ is called their **cross-ratio** and is written $\{AB, CD\}$.

[The cross-ratio $\{AB, CD\}$ is the ratio of the ratios in which C and D divide AB.]

We see that, if $\{AB, CD\} = -1$, A, C, B, D is a harmonic range.

Theorem 25.

If $\{AB, CD\} = -1$, then $\dfrac{1}{AC} + \dfrac{1}{AD} = \dfrac{2}{AB}$.

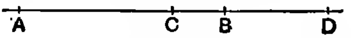

fig. 30.

Let $AB = x$, $AC = y$, $AD = z$.

$$\text{If } \{AB, CD\} = -1,$$

$$\text{then } \frac{AC}{CB}\Big/\frac{AD}{DB} = -1.$$

$$\therefore \frac{y}{x-y}\Big/\frac{z}{x-z} = -1.$$

$$\therefore \frac{y}{x-y} = -\frac{z}{x-z}.$$

$$\therefore yx - yz = -xz + yz.$$

$$\therefore xz + yx = 2yz.$$

$$\therefore \frac{1}{y} + \frac{1}{z} = \frac{2}{x}.$$

$$\text{i.e. } \frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}.$$

$\therefore$ AC, AB, AD are in harmonic progression; hence the name 'harmonic range.'

Ex. **212.** Prove that the same property is true for the distances measured from any one of the four points.

THEOREM 26.

If AB is divided harmonically at C, D, and if O is the mid-point of AB, then OC . OD = OB².

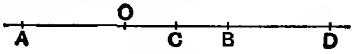

fig. 31.

Let OB $= b$, OC $= c$, OD $= d$; then AO $= b$.

$$\text{If } \{AB, CD\} = -1,$$

$$\text{then } \frac{AC}{CB} \Big/ \frac{AD}{DB} = -1.$$

$$\therefore \frac{b+c}{b-c} \Big/ \frac{b+d}{b-d} = -1.$$

$$\therefore \frac{b+c}{b-c} = -\frac{b+d}{b-d}.$$

$$\therefore b^2 + bc - bd - cd = -b^2 - bd + bc + cd.$$

$$\therefore 2b^2 = 2cd.$$

$$\therefore b^2 = cd.$$

$$\text{i.e. } OB^2 = OC . OD.$$

Ex. **213.** Prove the converse of the above proposition, namely, that if O is the mid-point of AB and OC . OD = OB², then $\{AB, CD\} = -1$.

Ex. **214.** If $\{AB, CD\} = -1$ and P is the mid-point of CD, then

$$PA . PB = PC^2.$$

Ex. **215.** If AB is divided harmonically at C, D and if O is the mid-point of AB and P of CD, prove that $OB^2 + PC^2 = OP^2$.

Ex. **216.** If $\{AB, CD\} = -1$, what is the position of D when C coincides with (i) A, (ii) the mid-point of AB, (iii) B, (iv) the point at infinity.

Ex. **217.** Prove that if ACBD be a harmonic range and if O be the middle point of CD, then AC is to CB as AO to CO.

Ex. **218.** P, Q divide a diameter of a circle harmonically; P′, Q′ divide another diameter harmonically; prove that P, P′, Q, Q′ are concyclic.

Ex. **219.** If X, Y, Z are the points at which the in-circle of a triangle ABC touches the sides, and if YZ produced cuts the opposite side in X′, then X and X′ divide that side harmonically.

[Use Menelaus' Theorem.]

Ex. **220.** Prove the same theorem for the points of contact of one of the ex-circles.

Ex. **221.** On a straight line take four points A, C, B, D such that AC=1·6 in., CB=0·8 in., BD=2·4 in. What is the value of {AB, CD}?

Take any point O outside the line. Draw a straight line parallel to OD cutting OA, OB, OC at P, Q, R. Find experimentally the value of the ratio PR/RQ.

Again draw parallels to OA, OB or OC in turn, and try to discover a law.

Ex. **222.** {AB, CD} = −1; O is any point outside the line ACBD; through C draw PCQ parallel to OD cutting OA, OB at P, Q. Prove PC=CQ

[By means of similar triangles express PC/OD in terms of segments of the line ACBD, and then express CQ/OD in the same way.]

Ex. **223.** Prove the converse of Ex. 222, namely, that if PC=CQ then {AB, CD} = −1.

Ex. **224.** Draw ACBD as in Ex. 221; take any point O outside the line and join OA, OC, OB, OD; draw a line cutting these lines at A′, C′, B′, D′; measure and calculate {A′B′, C′D′}. Repeat the experiment for another position of A′C′B′D′.

Ex. **225.** If a point O be joined to the points of a harmonic range ACBD and these lines be cut by a straight line in A′, C′, B′, D′; prove that {A′C′B′D′} is harmonic.

[Through C and C′ draw parallels to OD, and use Exs. 222, 223.]

Definition. A system of lines through a point is called a **pencil**. The point is called the **vertex** of the pencil.

Definition. Any straight line drawn across a system of lines is called a **transversal**.

THEOREM 27.

If a transversal cuts the four lines of a pencil at A, C, B, D, and if ACBD is a harmonic range, then any other transversal will also be divided harmonically.

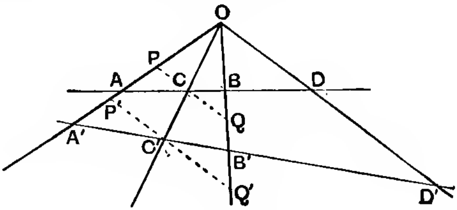

fig. 32.

Let O be the vertex of the pencil.

Let a straight line cut the rays of the pencil at A′, C′, B′, D′.

Through C and C′ draw straight lines ‖ to OD, cutting OA at P, P′ and OB at Q, Q′.

∵ {AB, CD} = −1
∴ PC = CQ

∵ PC is ‖ to OD.

∴ Δs APC, AOD are similar.

$$\therefore \frac{AC}{AD} = \frac{PC}{OD},$$

also Δs BQC, BOD are similar.

$$\therefore \frac{CB}{DB} = \frac{QC}{OD}.$$

Since {AB, CD} = −1,

$$\therefore \frac{AC}{CB} \Big/ \frac{AD}{DB} = -1.$$

$$\therefore \frac{AC}{AD} = -\frac{CB}{DB}.$$

$$\therefore \frac{PC}{OD} = -\frac{QC}{OD}.$$

∴ PC = CQ.

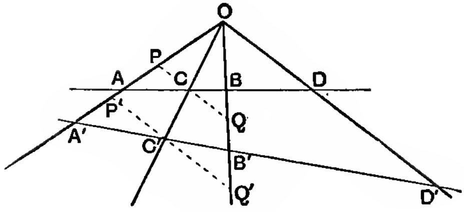

fig. 32.

$\therefore$ P′C′ = C′Q′

$$\text{Now } \frac{P'C'}{PC} = \frac{OC'}{OC} = \frac{C'Q'}{CQ}.$$

$$\therefore P'C' = C'Q'.$$

And from similar triangles as before,

$\therefore$ {A′B′, C′D′} = − 1

$$\frac{A'C'}{A'D'} = \frac{P'C'}{OD'},$$

$$\text{and } \frac{C'B'}{D'B'} = \frac{Q'C'}{OD'}.$$

$$\therefore \frac{A'C'}{A'D'} = -\frac{C'B'}{D'B'}.$$

$$\therefore \frac{A'C'}{C'B'} \Big/ \frac{A'D'}{D'B'} = -1.$$

$\therefore$ {A′B,′ C′D′} is a harmonic range.

Definition. If a pencil of four lines divides one transversal (and therefore every transversal*) harmonically, the pencil is called a **harmonic pencil.**

O{AB, CD} = − 1 denotes that the pencil OA, OB, OC, OD is harmonic; OC and OD are called harmonic conjugates with respect to OA and OB.

* This follows from the proposition just proved.

NOTE ON THEOREM 27.

From Theorem 26 it is easy to show that if C and D are harmonic conjugates with respect to AB, and if D is at infinity, then C is the mid-point of AB. [See also p. 8, § 3 (iii).]

In the course of proving Theorem 27 we saw that a transversal PQ parallel to OD is bisected by OC; it should be noticed that this is a particular case of the theorem; for, since PQ is parallel to OD it cuts OD at infinity; therefore C and the point at infinity are harmonic conjugates with respect to PQ; therefore C is the mid-point of PQ.

THEOREM 28.

The internal and external bisectors of an angle are harmonic conjugates with respect to the arms of the angle.

The proof is left to the reader.

THEOREM 29.

If $\{AB, CD\} = -1$ and O is a point outside the line such that $\angle$ COD is a right angle, then OC, OD are the bisectors of $\angle$ AOB.

The proof is left to the reader.

Ex. **226.** What line is the harmonic conjugate of the median of a triangle with respect to the two sides through the vertex from which the median is drawn?

Ex. **227.** If α, β, γ are the mid-points of the sides of a triangle ABC, prove that $\alpha\{\gamma\beta, AC\} = -1$.

Ex. **228.** If D, E, F are the feet of the altitudes of a triangle ABC, prove that $D\{EF, AB\} = -1$.

Ex. **229.** If X, Y, Z are the points of contact of the in-circle and the sides of the triangle ABC, prove that $X\{YZ, AC\} = -1$.

Ex. **230.** Lines are drawn parallel to the sides of a parallelogram through the intersection of its diagonals; prove that these lines and the diagonals form a harmonic pencil.

Ex. **231.** If A, B, C, D and A′, B′, C′, D′ are two harmonic ranges and if AA′, BB′, CC′ all pass through a point O, then O, D, D′ are collinear.

Exercises on Chapter VI.

Ex. **232.** A, B, C, D, O, P are points on a circle and $O\{AB, CD\} = -1$; prove that $P\{AB, CD\} = -1$.

Ex. **233.** The bisector of the angle A of a triangle ABC meets BC in X; prove that AX is divided harmonically by the perpendiculars drawn to it from B and C.

Ex. **234.** The pencil formed by joining the four angular points of a square to any point on the circumscribing circle of the square is harmonic.

Ex. **235.** A chord AB and a diameter CD of a circle cut at right angles. If P be any other point on the circle, P (AB, CD) is a harmonic pencil.

Ex. **236.** α, β, γ are the mid-points of the sides of a triangle ABC, $A\alpha$ and $\beta\gamma$ intersect at X, a line is drawn through X cutting $\alpha\gamma$, $\alpha\beta$, BC at Y, Z, W respectively; prove that Y, X, Z, W form a harmonic range.

Ex. **237.** Three lines pass through a point; through a given point on one of the lines draw a line that shall be divided into two equal parts by the other two.

Ex. **238.** Find a point P in a given straight line so that the lines joining P to three given points in a plane containing the given line may cut off on any line parallel to the given line and lying in the same plane two equal segments.

Ex. **239.** **If X, Y, Z are points on the sides BC, CA, AB of a triangle such that AX, BY, CZ are concurrent, and if YZ meets BC in X′; then is {BC, XX′} a harmonic range.**

Ex. **240.** TP, TQ are two tangents to a circle; PR is a diameter of the circle and QN is drawn perpendicular to PR. Prove that Q{TPNR} is a harmonic pencil.

Ex. **241.** In a triangle ABC the line AD is drawn bisecting the angle A and meeting BC in D. Find a point P in BC produced either way, such that the square on PD may be equal to the rectangle PB . PC.

Ex. **242.** P is a point on the same straight line as the harmonic range ABCD; prove that

$$2\frac{PA}{AC}=\frac{PB}{BC}+\frac{PD}{DC}.$$

Ex. **243.** A, B, C, D are four points in a straight line; find two points in the line which are harmonic conjugates with respect to A, B and also with respect to C, D.

Ex. **244.** ABC is a triangle; through D, the mid-point of BC, a straight line PDQR is drawn cuttiog AB, AC in P, Q respectively. AR is drawn parallel to BC, and cuts BQ at S. Prove that AR = RS.

Ex. **245.** PAQB is a harmonic range, and a circle is drawn with AB as diameter. A tangent from P meets the tangent at B in S, and touches the circle in T. Prove that SA bisects TQ.

Ex. **246.** Through one angle O of a parallelogram OEAF a line is drawn meeting AE and AF, both produced, in B and C respectively. Prove that the area AEOF is a harmonic mean between the areas BOA and COA.

Ex. **247.** TP, TQ are two tangents to a circle; PR is a diameter of the circle and QN is drawn perpendicular to PR. Prove that QN is bisected by TR.

Ex. **248.** If X is any point in AD an altitude of a triangle ABC, and BX, CX produced cut the opposite sides of the triangle in Y and Z, then $\angle$ YDZ is bisected by DA.

Ex. **249.** Prove that the lines joining any point on a circle to the ends of a fixed chord cut the diameter perpendicular to the chord in two points which divide the diameter harmonically.

Ex. **250.** If A′, B′, C′ lie on the sides BC, CA, AB of a triangle and AA′, BB′, CC′ be concurrent; and if A″ be the harmonic conjugate of A′ with respect to B, C while B″, C″ are similarly determined on the other sides; then A″, B″, C″ are collinear.

Ex. **251.** The lines VA′, VB′, VC′ bisect the internal angles formed by the lines joining any point V to the angular points of the triangle ABC; and A′ lies on BC, B′ on CA, C′ on AB. Also A″, B″, C″ are harmonic conjugates of A′, B′, C′ with respect to B and C, C and A, A and B. Prove that A″, B″, C″ are collinear.

Ex. **252.** The inscribed circle of a triangle ABC touches the sides BC, CA, AB in A′, B′, C′ respectively. Show that the points in which the circles B′C′B and B′C′C meet BC again are equidistant from A′.

CHAPTER VII.

POLE AND POLAR.

Definition. The line joining the points at which two tangents touch a circle is called their **chord of contact.**

Ex. **253.** If tangents are drawn to a circle from an external point, the line joining this point to the centre of the circle bisects the chord of contact at right angles.

Ex. **254.** What is the chord of contact of a point on the circumference?

Provisional Definition. If P and Q are the points of contact of the tangents to a circle from a point T, the straight line through P and Q is called the **polar** of T, and T is called the **pole** of PQ, with respect to the circle.

This definition of the polar of a point is meaningless when the point is inside the circle. It will therefore be necessary to find a new definition. But before doing so we must prove the following theorem.

THEOREM 30.

If the line joining a point T to the centre C of a circle cuts the cnord of contact of T in N and the circle in A, then $CN . CT = CA^2$.

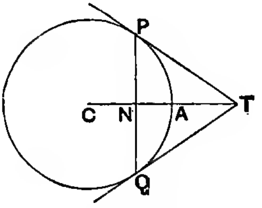

fig. 33.

Let P and Q be the points of contact of the tangents from T.

Then, in the $\triangle$s CPN, CTP, $\angle$ C is common,

and $\angle$s CNP, CPT are equal (being rt. $\angle$s).

$\therefore$ the $\triangle$s are similar.

$$\therefore \frac{CN}{CP} = \frac{CP}{CT}.$$

$$\therefore CN . CT = CP^2 = CA^2.$$

Definition. If T and N are two points on a line, drawn from C the centre of a circle, such that CN . CT is equal to the square on a radius of the circle, and if through N a line XY is drawn at right angles to CN, XY is called the **polar** of T and T is called the **pole** of XY with respect to the circle.

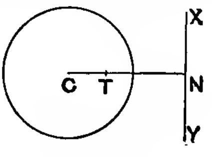

fig. 34.

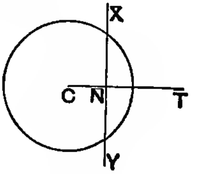

fig. 35.

Ex. **255.** Prove that this definition agrees with the 'provisional definition' when T is outside the circle.

Ex. **256.** What is the position of the polar in the following cases: (i) T on the circle, (ii) T coinciding with C, (iii) T at infinity?

Ex. **257.** A and B are two concentric circles; what is the envelope of the polar with respect to A of a point which moves round B?

Ex. **258.** What are the polars of the vertices of a triangle with respect to (i) its incircle, (ii) its circumcircle?

Ex. **259.** ABC is a triangle. A circle is described with A as centre and radius AX such that $AX^2 = AB . AF$ where F is the altitude from C. What lines are the polars of B and C? and what point is the pole of BC?

Ex. **260. If from a fixed point T any line is drawn cutting a circle in R and S, prove that the tangents at R and S intersect on a fixed line (viz. the polar of T).**

[Let X be the point of intersection of the tangents; draw XN ⊥ to the line joining T to the centre C; let CX cut RS in K. Prove that $CN . CT = CK . CX = CS^2$.]

Ex. **261.** Prove that any point T and the polar of T with respect to a circle divide the diameter through T harmonically.

Theorem 31.

If a straight line is drawn through any point to cut a circle, the line is divided harmonically by the circle, the point and the polar of the point with respect to the circle.

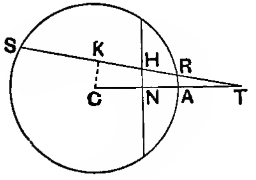

fig. 36.

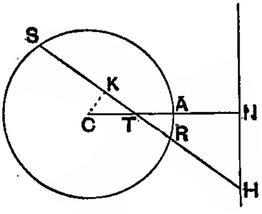

fig. 37.

Let T be the point, TRHS the line cutting the circle at R, S and the polar of T at H; let C be the centre of the circle, and let CT cut the circle at A and the polar of T at N.

Draw CK $\perp$ to RS; then K is the mid-point of RS.

[If we can prove $KH . KT = KR^2$, then $\{RS, HT\} = -1$.]

Sense being taken into account, we see that

$$KH . KT = KT (KT - HT)$$
$$= KT^2 - KT . HT.$$

Now in both figures C, K, H, N are concyclic, because the $\angle$s at K and N are right angles.

$$\therefore \quad KT . HT = CT . NT.$$
$$\therefore \quad KH . KT = KT^2 - CT . NT$$
$$= KT^2 - CT (CT - CN)$$
$$= KT^2 - CT^2 + CN . CT$$
$$= - CK^2 + CA^2 \qquad (CN . CT = CA^2 \text{ by def.})$$
$$= CR^2 - CK^2$$
$$= KR^2.$$

$\therefore$ $\{RS, HT\}$ is harmonic.

Ex. **262.** **If H be the harmonic conjugate of a fixed point T with regard to the points in which a line through T cuts a fixed circle, the locus of H is a straight line.**

[Use reductio ad absurdum.]

Ex. **263.** If C be the centre of a circle, and the polar of a point T cut TC in N, and any straight line through T cut the circle in R and S, then the polar bisects the angle RNS.

Ex. **264.** If a straight line TRS cut a circle in R and S and cut the polar of T in H, and if K be the mid-point of RS, then $TR . TS = TH . TK$.

Ex. **265.** The polar of a point O with regard to a circle meets it in A, B; any chord through O meets the circle in C, D. Prove that A, B, C, D subtend a harmonic pencil at any point of the circle.

[Consider the pencil subtended at A.]

Ex. **266.** A chord PQ of a circle moves so that the angle it subtends at a fixed point O inside the circle is bisected externally by the diameter through O; prove that PQ passes through a fixed point.

Is this theorem true when O is the centre of the circle?

Ex. **267.** State and prove a theorem corresponding to Ex. 266 for the case in which the diameter bisects the angle internally, O being still inside the circle.

Ex. **268.** A chord PQ of a circle moves so that the angle it subtends at a fixed point O outside the circle is bisected by the diameter which passes through O; prove that PQ either passes through a fixed point or is parallel to a fixed direction.

Ex. **269.** Prove that, if the points A, B, C, D all lie on a circle, the polar of the point of intersection of AC, BD passes through the point of intersection of AB, CD.

[Let AB, CD intersect at X, and AC, BD at Y; find Z the harmonic conjugate of Y with respect to B, D; let CA meet ZX in T; prove XT the polar of Y.]

Theorem 32.

If the polar of a point P with respect to a circle passes through a point Q, then the polar of Q passes through P.

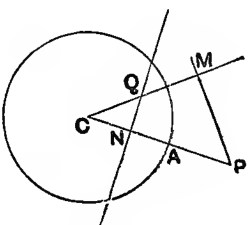

fig. 38.

Let C be the centre of the circle, and QN the polar of P.

From P draw PM $\perp$ to CQ.

Then Q, M, N, P are concyclic.

$\therefore$ CM . CQ = CN . CP

= CA^2. ($\because$ QN is the polar of P)

$\therefore$ PM is the polar of Q.

$\therefore$ the polar of Q passes through P.

Ex. **270.** Sketch a figure for Theorem 32 with both P and Q outside the circle.

Ex. **271.** Prove this theorem by the harmonic property of pole and polar for the particular case in which PQ cuts the circle.

Ex. **272.** **If a point moves on a straight line its polar with respect to a circle passes through a fixed point.**

Ex. **273.** **If a straight line moves so that it always passes through a fixed point, its pole with respect to a circle moves on a straight line.**

Ex. **274.** **The line joining any two points A and B is the polar of the point of intersection of the polars of A and B.**

Ex. **275.** AB, AC touch a circle at B, C. If the tangent at any other point P cuts BC produced at Q, prove that Q is the pole of AP.

Ex. **276.** AB, AC touch a circle at B, C; the tangent at another point P on the circle cuts BC at Q. Prove that A{BC, PQ} = −1.

THEOREM 33.

Two tangents are drawn to a circle from a point A on the polar of a point B; a harmonic pencil is formed by the two tangents from A, the polar of B and the line AB.

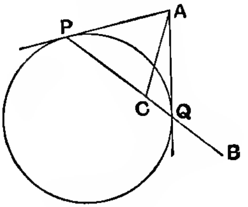

fig. 39.

Let AP, AQ be the tangents from A.

Since the polar of B passes through A, ∴ the polar of A (i.e. PQ) passes through B.

Let the polar of B cut PQ at C.

Then P, C, Q, B is a harmonic range. [Th. 31.]

∴ the pencil AP, AC, AQ, AB is harmonic.

Ex. **277.** Prove Ex. 219 by means of Theorem 33.

Ex. **278.** **From any point P on a fixed straight line XY tangents PZ, PW are drawn to a circle; prove that, if PT is such that the pencil P{ZW, YT} is harmonic, PT passes through a fixed point.**

[Prove that the intersection of ZW, PT is the pole of XY.]

Ex. **279.** Prove that, if the lines PX, PY, QX, QY all touch a circle, then XY passes through the pole of PQ.

[Draw PZ to cut XY at Z, such that P{XY, ZQ} = −1; and consider the pencil Q{XY, ZP}.]

An interesting case of pole and polar is that in which the circle has an infinite radius.

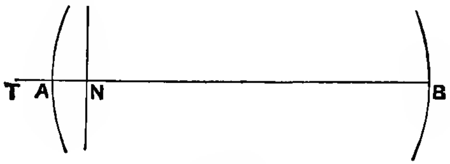

fig. 40.

Let AB be a diameter of a circle, T any point on it and let the polar of T cut AB at N; then TN is divided harmonically at A and B.

Now suppose that A and T remain fixed and that B moves along the line TA further and further from B; in the limit when B has moved to an infinite distance, TA = AN (since $\{TN, AB\} = -1$); and the circle becomes the line at infinity together with the line through A at right angles to TN.

Thus the polar of a point T with respect to a line (regarded as part of a circle of infinite radius) is a parallel line whose distance from T is double the distance of the given line from T.

Ex. **280.** Into what do the following properties degenerate in the case in which the circle has an infinite radius: (i) Theorem 31, (ii) Ex. 262, (iii) Ex. 272, (iv) Ex. 273?

Exercises on Chapter VII.

Ex. **281.** Through a point A within a circle are drawn two chords PP′, QQ′; show that PQ, P′Q′ subtend equal angles at B, the foot of the perpendicular from A to the polar of A with respect to the circle.

Ex. **282.** TP, TQ are two tangents to a circle; prove that the tangent to the circle from any point on PQ produced is divided harmonically by TP and TQ.

Ex. **283.** The tangents at two points P and Q of a circle intersect at T; HTK is drawn parallel to the tangent at a point R, and meets PR and QR in H and K respectively; prove that HK is bisected in T.

Ex. **284.** From a point O a line is drawn cutting a circle in P and R and the polar of O in Q; if N is the mid-point of PR and if the polar of O meets the circle in T and T′, show that the circles TQN, T′QN touch OT, OT′ respectively.

Ex. **285.** A fixed point A is joined to any point P on a circle, AQ is drawn to cut the tangent at P in Q so that $\angle PAQ = \angle APQ$, AQ is produced to R and $QR = AQ$; prove that R lies on the polar of A.

Ex. **286.** From a point O are drawn two straight lines, OT to touch a given circle at T and OC to pass through its centre C, and TN is drawn to cut OC at right angles in N. Show that the circle which touches OC at O and passes through T cuts the given circle at a point S such that the straight line TS produced bisects NO.

Ex. **287.** AOB, COD are chords of a circle intersecting in O. The tangents at A and D meet in P, and the tangents at B and C meet in Q. Show that P, O, Q are collinear.

Ex. **288.** The product of the perpendiculars on any two tangents to a circle from any point on its circumference is equal to the square on the perpendicular from the point to the chord of contact.

Ex. **289.** I is the centre of the incircle of a triangle ABC; lines through I perpendicular to IA, IB, IC meet the tangent at P to the incircle in D, E, F respectively. Find the positions of the poles of AD, BE, CF with respect to the incircle: and hence (or otherwise) prove that these three lines are concurrent.

Ex. **290.** The distances of two points from the centre of a circle are in the same ratio as their distances each from the polar of the other with respect to the circle (Salmon's theorem).

Ex. **291.** The harmonic mean of the perpendiculars from any point O within a circle to the tangents drawn from any point on the polar of O is constant.

CHAPTER VIII.

SIMILITUDE.

1. In elementary geometry* we have seen that, if a point O is joined to each vertex of a given polygon, and if each of the joins is divided in the same ratio, these points of division are the vertices of a similar polygon.

Extending this principle, we see that, *if a point* O *is joined to a point* P, *and* OP *is divided in a fixed ratio at* Q, *as* P *describes a given figure (consisting of any number of lines and curves), the point* Q *will describe a similar figure.*

Ex. **292.** Draw a circle of radius 4 cm.; mark a point O, 10 cm. from its centre; if P is any point on the circle plot the locus of the mid-point of OP.

Ex. **293.** Prove that the locus is a circle in Ex. 292.

Ex. **294.** P is a variable point on a fixed circle whose centre is O; a point Q is taken on the tangent at P, such that angle POQ is constant; what is the locus of Q?

Ex. **295.** Draw a triangle ABC having BC=8 cm., CA=6 cm., AB=7 cm.; mark a point P 4 cm. from B and 6 cm. from C. The triangle is now rotated about P through a right angle, to the position *abc*; explain how you determine the points *a*, *b*, *c* and find what angle *ac* makes with AC.

2. *If a figure is rotated about a point* O *through any angle* α, *the angle through which any line in the figure has been rotated (i.e. the angle between the new position and the old) is* α.

* Godfrey and Siddons' *Elementary Geometry*, IV. 9.

3. *Again suppose* O *a fixed point and* P *any point on a given figure, and* Q *a point such that* $OQ : OP = k : 1$, *and* $\angle POQ = \alpha$ (k *and* α *being constants*); *as* P *describes any figure,* Q *will describe a similar figure.*

For suppose Q′ a point in OP such that $OQ' : OP = k : 1$, Q′ will describe a figure similar to the "P" figure; and if we now rotate the "Q′" figure about O through an angle α it will coincide with the "Q" figure.

Ex. **296.** P is a variable point on a fixed circle, O any point inside it; PQ is drawn at right angles to OP and OQ makes a fixed angle (always taken in the same sense) with OP. What is the locus of Q?

4. *If* ABC, DEF *are two similar triangles with their corresponding sides parallel, then* AD, BE, CF *will be concurrent.*

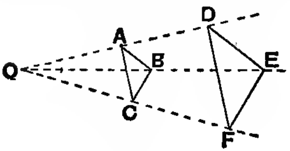

fig. 41.

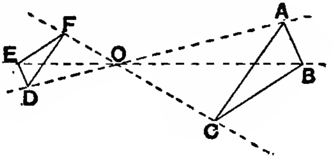

fig. 42.

For if AD and BE cut at O, $OA : OD = AB : DE$; and if AD and CF cut at O′, $O'A : O'D = AC : DF = AB : DE = OA : OD$; $\therefore$ O and O′ coincide.

Extending this we see that, *if* ABCD ... A′B′C′D′ ... *are two similar rectilinear figures with their corresponding sides parallel,* AA′, BB′, CC′, DD′, ... *are concurrent;* or going a step further we see that *the same is true even when the figures consist of curves as well as straight lines.*

When two similar figures are so placed that the join of each pair of points in the one figure is parallel to the join of the corresponding pair of points in the other figure, the two figures are said to be **similarly situated** and the point of concurrence of the lines joining corresponding points is called the **centre of similitude.**

In the case of triangles we have seen that AD is divided (externally or internally) in the ratio of AB : DE.

So in the general case the centre of similitude divides the joins of corresponding points in the ratio of the linear dimensions of the two figures.

Ex. **297.** Draw a careful figure of two similar and similarly situated circles; indicate several corresponding points and draw the tangents at a pair of such points.

Ex. **298.** Draw (or plot) an accurate parabola. Draw a similar and similarly situated curve, (i) when the centre of similitude is on the axis, (ii) when it is not on the axis.

Ex. **299.** **If any line through the centre of similitude of two curves cuts them at corresponding points P and P′, the tangents at P and P′ are parallel.**

Or in other words, **any line through the centre of similitude of two curves cuts them at the same angle at corresponding points.**

[Use the method of limits.]

Ex. **300.** **If O is a centre of similitude of two curves, a tangent from O to either of the curves touches the other curve, and the points of contact are corresponding points.**

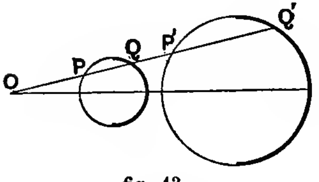

fig. 43.

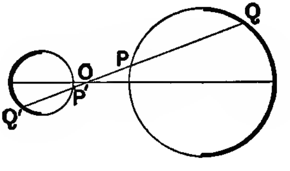

fig. 44.

5. *In the case of two circles there are two centres of similitude, viz. the points which divide the line joining the centres externally and internally in the ratio of the radii.*

In fig. 44 the constant ratio is negative.

Ex. **301.** **If a circle A touches two circles B, C at points P, Q, prove that PQ passes through a centre of similitude of the circles B, C.**

Note that there are two cases.

Ex. **302.** **Prove that the common tangents to two circles pass through one of their centres of similitude.**

Ex. **303.** What is the centre of similitude of a line and a circle? Have they two centres of similitude?

Ex. **304.** Have two parallel lines a centre of similitude?

Ex. **305.** Have two intersecting lines a centre of similitude?

Exercises on Chapter VIII.

Ex. **306.** A triangle ABC is given *in specie* (i.e. its angles are given) and the point A is fixed; prove that B and C describe similar loci.

Ex. **307.** Find the centres of similitude of the circumcircle and nine-points circle of a triangle.

Ex. **308.** O is a fixed point, XOY a constant angle, $\frac{OX}{OY}$ a constant ratio. Find the locus of Y when that of X is (i) a straight line, (ii) a circle.

Ex. **309.** Prove that the line joining the vertex of a triangle to that point of the inscribed circle which is furthest from the base passes through the point of contact of the escribed circle with the base.

Ex. **310.** A triangle ABC is inscribed in a given circle, and its vertex A is fixed. Show that the locus of a point P on BC, such that the ratio of AD^2 to BD . DC is given, is a circle touching the given circle at A.

Ex. **311.** C is a moving point on a circle of which O is centre and AB is a fixed diameter; BC is produced to D so that BC=CD. Find the locus of the intersection of AC and OD.

Ex. **312.** In a quadrilateral ABCD, the points A and B are fixed, and the lengths BC, CA and CD are given. Find the locus of (1) the mid-point of BD, (2) the mid-point of the line joining the mid-points of the diagonals.

Ex. **313.** Through a point O draw a line cutting a circle in P, Q, such that the chord PQ is $\frac{1}{3}$ of OQ.

Ex. **314.** A is a moving point on a fixed diameter BD (produced) of a circle; AC is a tangent from A; P is the projection of the centre on the bisector of the angle BAC. Find the locus of P.

Ex. **315.** Inscribe in an equilateral triangle another equilateral triangle having each side equal to a given straight line.

Ex. **316.** Describe a triangle of given species (given angle) so that one angular point may be at a given point and the others on given straight lines.

Ex. **317.** O is a fixed point, and a straight line OPQ revolving round O cuts a fixed circle in P, Q. On this line is a point R such that $OP \cdot OR = k^2$. Find the locus of R.

CHAPTER IX.

MISCELLANEOUS PROPERTIES OF THE CIRCLE.

SECTION I. ORTHOGONAL CIRCLES.

Definition. The **angles at which two curves intersect** are the angles between the tangents to the curves at their point of intersection.

Ex. **318.** **If two circles intersect at P and Q, the angles at which they intersect at P are equal to the angles at which they intersect at Q.**

Definition. When two circles intersect at right angles, they are said to intersect **orthogonally** and are called **orthogonal circles.**

THEOREM 34.

If two circles are orthogonal, a tangent to either at their point of intersection passes through the centre of the other.

The proof is left to the reader.

Ex. **319.** Prove Theorem 34.

Ex. **320.** Two circles A and B are orthogonal if the tangent to A from the centre of B is equal to the radius of B.

Ex. **321.** Through two given points on a circle draw a circle to cut the given circle orthogonally.

Is this always possible?

Ex. **322.** Through a given point on a circle draw a circle of given radius to cut the given circle orthogonally.

Is this always possible?

Ex. **323.** The tangents drawn from a point P to two circles are equal; prove that a circle can be described with P as centre to cut both circles orthogonally.

Ex. **324.** The pole of the common chord of two orthogonal circles with respect to one of the circles is the centre of the other.

THEOREM 35.

The sum of the squares on the radii of two orthogonal circles is equal to the square on the distance between their centres.

The proof is left to the reader.

Ex. **325.** Prove Theorem 35.

Ex. **326.** State and prove the converse of Theorem 35.

Ex. **327.** If two circles be described upon the straight lines joining the two pairs of conjugate points of a harmonic range as diameters, the circles cut orthogonally.

THEOREM 36.

Any diameter of a circle which cuts an orthogonal circle is divided harmonically by the orthogonal circle.

The proof is left to the reader.

Ex. **328.** Prove Theorem 36.

Ex. **329.** **If P, Q divide a diameter of a given circle harmonically, any circle through PQ cuts the given circle orthogonally.**

Ex. **330.** A variable circle passes through a fixed point and cuts a given circle orthogonally; prove that the variable circle passes through another fixed point.

Ex. **331.** Describe a circle to cut a given circle orthogonally and pass through two given points.

Is this always possible?

Ex. **332.** If a pair of orthogonal circles intersect at P and Q, and if the line APB cuts the circles at A and B, then AB subtends a right angle at Q.

Ex. **333.** Circles are orthogonal if the angles in the major segments on opposite sides of the chord of intersection are complementary.

Ex. **334.** The locus of the points of intersection of the straight lines joining two fixed points on a circle to the extremities of a variable diameter is the circle through the fixed points orthogonal to the given circle.

SECTION II. THE CIRCLE OF APOLLONIUS*.

THEOREM 37.

If a point P moves so that the ratio of its distances from two fixed points Q, R is constant, the locus of P is a circle.

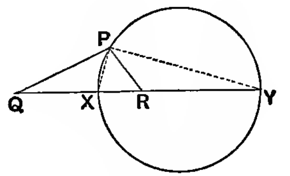

fig. 45.

For any position of P draw PX, PY, the bisectors of the angle QPR, to cut QR in X, Y respectively.

Since PX bisects $\angle$ QPR,

$\therefore$ QX : XR = QP : PR

= the given ratio.

$\therefore$ X is a fixed point.

Similarly Y is a fixed point.

* See note on p. 20.

Again, since PX, PY are the bisectors of $\angle$ QPR,

$\therefore$ $\angle$ XPY is a right angle.

$\therefore$ the locus of P is the circle on XY as diameter.

Ex. **335.** Construct a triangle having given its base, the ratio of its other two sides and its area.

Ex. **336.** Construct a triangle having given one side, the angle opposite to that side and the ratio of the other two sides.

Ex. **337.** Find a point such that its distances from three given points are in given ratios.

How many solutions are there?

Ex. **338.** Given the ratio of the two sides of a triangle, the middle point of the third side, the point in which this side is met by the bisector of the angle opposite to it and the direction of this bisector, construct the triangle.

Ex. **339.** In fig. 45 prove that the tangent at P passes through the circumcentre of the triangle PQR.

Ex. **340.** The internal and external bisectors of the angles of a triangle are drawn, and on the lengths they intercept on the opposite sides circles are described having these intercepts as diameters; prove that these circles will all pass through two points.

Section III. Ptolemy's* Theorem.

Theorem 38.

The sum of the rectangles contained by opposite sides of a cyclic quadrilateral is equal to the rectangle contained by its diagonals.

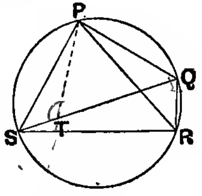

fig. 46.

Let PQRS be the quadrilateral.

Make $\angle$ SPT $=$ $\angle$ RPQ, and let PT cut SQ at T.

Now $\triangle$s SPT, RPQ are equiangular

$$(\angle \text{SPT} = \angle \text{RPQ},\ \angle \text{PST} = \angle \text{PRQ}),$$
$$\therefore\ \text{PS} : \text{PR} = \text{ST} : \text{RQ},$$
$$\therefore\ \text{PS} . \text{RQ} = \text{PR} . \text{ST}.$$

Again $\triangle$s TPQ, SPR are equiangular

$$(\angle \text{TPQ} = \angle \text{SPR},\ \angle \text{PQT} = \angle \text{PRS}),$$
$$\therefore\ \text{PQ} : \text{PR} = \text{TQ} : \text{SR},$$
$$\therefore\ \text{PQ} . \text{SR} = \text{PR} . \text{TQ},$$
$$\therefore\ \text{PS} . \text{RQ} + \text{PQ} . \text{SR} = \text{PR} . \text{ST} + \text{PR} . \text{TQ}$$
$$= \text{PR} . \text{SQ}.$$

Ex. **341.** What does Ptolemy's theorem become in the special case in which two vertices of the quadrilateral coincide?

Ex. **342.** What does Ptolemy's theorem become in the special case in which the circle becomes a straight line?

Prove the theorem independently.

Ex. **343.** ABC is an equilateral triangle inscribed in a circle; P is any point on the minor arc BC. Prove that PA $=$ PB $+$ PC.

* Ptolemy was a great Greek astronomer, and one of the earliest writers on trigonometry (87—165 A.D.).

THEOREM 39.

The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by its opposite sides unless the quadrilateral is cyclic, in which case it is equal to that sum.

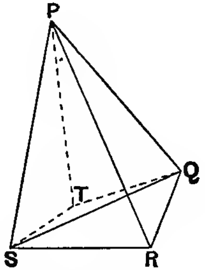

fig. 47.

Let PQRS be the quadrilateral.
Make $\angle SPT = \angle RPQ$ and $\angle PST = \angle PRQ$.
Now Δs SPT, RPQ are equiangular by construction,

$$\therefore PS : PR = ST : RQ,$$
$$\therefore PS . RQ = PR . ST.$$
$$\text{Also } PT : PQ = PS : PR,$$
$$\therefore PT : PS = PQ : PR$$
$$\text{and } \angle TPQ = \angle SPR,$$
$$\therefore \Delta\text{s } TPQ, SPR \text{ are equiangular},$$
$$\therefore PQ : PR = TQ : SR,$$
$$\therefore PQ . SR = PR . TQ,$$
$$\therefore PR . ST + PR . TQ = PS . RQ + PQ . SR.$$

But $SQ < ST + TQ$ unless STQ is a straight line.
$\therefore PR . SQ < PS . RQ + PQ . SR$ unless STQ is a straight line.
If STQ is a straight line,

$$\angle QSP = \angle QRP \text{ by construction.}$$

$\therefore$ in that case P, Q, R, S are concyclic.

Note that this theorem includes the converse of Ptolemy's theorem

Ex. **344.** If ABC is an equilateral triangle, find the locus of a point which moves so that the sum of its distances from B and C is equal to its distance from A.

Several theorems in trigonometry may be proved by means of Ptolemy's theorem, but of course the proofs do not apply to angles greater than two right angles.

As an example, we will prove that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

In fig. 48, let PR be a diameter and $\angle$ SPR $= \alpha$, $\angle$ RPQ $= \beta$, and let ρ be the radius of the circle.

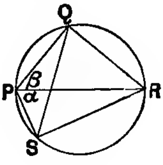

fig. 48.

Then PQ $= 2\rho\cos\beta$, RQ $= 2\rho\sin\beta$, SR $= 2\rho\sin\alpha$, PS $= 2\rho\cos\alpha$, PR $= 2\rho$. Also by Th. 5 SQ $= 2\rho\sin(\alpha+\beta)$.

By Ptolemy's theorem

$$\text{PR . SQ} = \text{PS . RQ} + \text{PQ . SR},$$

$$\therefore\ 2\rho\,.\,2\rho\sin(\alpha+\beta) = 2\rho\cos\alpha\,.\,2\rho\sin\beta + 2\rho\cos\beta\,.\,2\rho\sin\alpha,$$

$$\therefore\ \sin(\alpha+\beta) = \cos\alpha\sin\beta + \cos\beta\sin\alpha.$$

Ex. **345.** Prove the formula for $\cos(\alpha+\beta)$ by taking PQ a diameter, $\angle$ QPR $= \alpha$, and $\angle$ PQS $= \beta$.

Ex. **346.** Prove the formulae for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$.

SECTION IV. CONTACT PROBLEMS.

Consider the problem of describing a circle to touch three given circles. As particular cases any of the three circles may become a line or point.

For the sake of clearness it will be convenient to adopt abbreviations in Exs. 347—357, e.g. "Describe a circle having given $P_2L_1C_0$" will be used as an abbreviation for "Describe a circle to pass through two given points and touch a given line."

Ex. 347. Show that ten different cases may arise out of this.

Ex. 348. State two cases which are already familiar. How many solutions are there in each case?

Ex. 349. Describe a circle having given $P_2L_1C_0$.

How many solutions are there?

[Produce the line joining the two points to cut the given line; where will the point of contact be?]

Ex. 350. Describe a circle having given $P_2L_0C_1$.

How many solutions are there?

[Draw any circle through the two points to cut the given circle; let their radical axis meet the line joining the two points in T; draw tangents from T.]

Ex. 351. Describe a circle having given $P_1L_2C_0$.

How many solutions are there?

[Describe any circle touching the two lines and magnify it.]

Ex. **352.** If a circle touches a line and a circle, the line joining the points of contact passes through one end of the diameter at right angles to the given line.

Note that the ends of the diameter are the centres of similitude of the line and circle.

Ex. 353. Describe a circle having given $P_1L_1C_1$.

How many solutions are there?

[See Ex. 352; let A, B be the ends of the diameter, and let AB cut the line in C; let M, N be the points of contact, P the given point, and let AP cut the required circle in P′; then $AB \cdot AC = AM \cdot AN = AP \cdot AP'$.]

Ex. 354. Describe a circle having given $P_1L_0C_2$.

How many solutions are there?

[Take a centre of similitude of the two circles, and see note to Ex. 353.]

Ex. 355. Describe a circle having given $P_0L_2C_1$.

How many solutions are there?

[Move the lines parallel to themselves through a distance equal to the radius of the circle; describe a circle to touch these lines and pass through the centre of the given circle; this circle will be concentric with the required circle.

This process is called **the method of parallel translation.**]

Ex. 356. Describe a circle having given $P_0L_1C_2$.

How many solutions are there?

[Reduce one of the circles to a point by the method of parallel translation.]

Ex. 357. Describe a circle having given $P_0L_0C_3$.

[Use the method of parallel translation.]

Exercises on Chapter IX.

Ex. **358.** Prove that the locus of the centres of circles passing through a given point and cutting a given circle orthogonally is a straight line.

Ex. **359.** Show that, if AB is a diameter of a circle which cuts two given circles orthogonally, the polars of A with respect to the two circles intersect in B.

Ex. **360.** O is a common point of two orthogonal circles, A, A′ are the points of contact of one common tangent, B, B′ of the other.

Show that one of the angles AOA′, BOB′ is half a right angle and that their sum is two right angles.

Ex. **361.** Two fixed circles intersect in A, B; P is a variable point on one of them; PA meets the other circle in X and PB meets it in Y. Prove that BX and AY intersect on a fixed circle.

Ex. **362.** Find the locus of the points at which two given circles subtend the same angle.

Ex. **363.** If A, B be two fixed points in a fixed plane, and P a point which moves in the plane so that $AP=m.BP$, where $m>1$, show that P describes a circle whose radius is $\frac{m.AB}{m^2-1}$.

Show also that if two tangents to the circle be drawn from A, their chord of contact passes through B.

Ex. **364.** Four points A, B, A′, B′ are given in a plane; prove that there are always two positions of a point C in the plane such that the triangles CAB, CA′B′ are similar, the equal angles being denoted by corresponding letters.

Ex. **365.** Three chords AA′, BB′, CC′ of a circle are concurrent. Show that the product of the lengths of the chords AB′, BC′, CA′ is equal to that of the chords BA′, CB′, AC′.

Ex. **366.** Show that a line cannot be divided harmonically by two circles which cut orthogonally, unless it passes through one or other of the centres.

Ex. **367.** The bisectors of the angles A, B, C of a triangle cut the opposite sides in X_1, X_2; Y_1, Y_2; Z_1, Z_2 respectively.

Show that the circles on the lines X_1X_2, Y_1Y_2, Z_1Z_2 as diameters have a common chord.

Ex. **368.** Construct a triangle, having given the length of the internal bisector of one angle, the ratio of the side opposite that angle to the sum of the other sides, and the difference of the other angles.

Ex. **369.** It is required to draw a circle to touch two given straight lines and a given circle. Prove that the eight possible points of contact with the circle may be found thus:—

Draw tangents to the circle parallel to the two lines and join the vertices of the rhombus so formed to the point of intersection of the two lines.

These lines cut the circle in the required points.

Ex. **370.** Describe a circle:

(i) to touch a given line and pass through two given points,

(ii) to pass through two given points and cut off from a given line a chord of given length.

(iii) to pass through two given points, so that the tangent drawn to it from another given point may be of given length.

Ex. **371.** Two circles, centres A and B, intersect at right angles at Q and Q′. A line PQR cuts the circles again at P and R. Show that AB subtends a right angle at the middle point of PR.

Ex. **372.** From a given point O, straight lines OA, OB, OC are drawn cutting a fixed straight line in A, B, C. A circle OBD is described cutting the circle OAC orthogonally, D being a point on the straight line ABC. Prove that either the angles AOB and COD are complementary, or one of these angles and the supplement of the other are complementary.

Ex. **373.** On a given chord AB of a circle, a fixed point C is taken, and another chord EF is drawn so that the lines AF, BE, and the line joining C to the middle point of EF meet in a point O; show that the locus of O is a circle.

Ex. **374.** A straight line is drawn cutting the sides BC, CA, AB of the triangle ABC in the points D, E, F respectively, so that the ratio FD to DE is constant; show that the circles FBD, CDE pass through a fixed point.

Ex. **375.** If S, S′ are the centres of similitude of two circles, prove that the circles subtend equal angles at any point on the circle whose diameter is SS′.

Ex. **376.** Construct a quadrilateral given the two diagonals, the angle at which they cut, and a pair of opposite angles.

Ex. **377.** A variable circle passes through a fixed point C and is such that the polar of a given point A with respect to it passes through a fixed point B; show that the locus of the centre of the circle is a straight line perpendicular to that joining C to the middle point of AB.

Ex. **378.** If two sides of a triangle of given shape and size always pass through two fixed points, the third side always touches a fixed circle.

[The centre of this circle lies on the locus of the vertex of the triangle, and its radius is equal to an altitude of the triangle.]

Ex. **379.** If two sides of a triangle of given shape and size slide along two fixed circles, the envelope of the third side is a circle. [Bobillier's Theorem.]

CHAPTER X.

THE RADICAL AXIS; COAXAL CIRCLES.

Ex. **380.** Draw a pair of circles intersecting at points P and Q; from any point on PQ produced draw tangents to the circles; prove that these tangents are of equal length.

Definition. The locus of the points from which tangents drawn to two circles are equal is called the **radical axis** of the two circles.

In Ex. 380, we have seen that in the case of two intersecting circles any point on their common chord produced is on their radical axis. This is a particular case of the following theorem.

THEOREM 40.

The radical axis of two circles is a straight line.

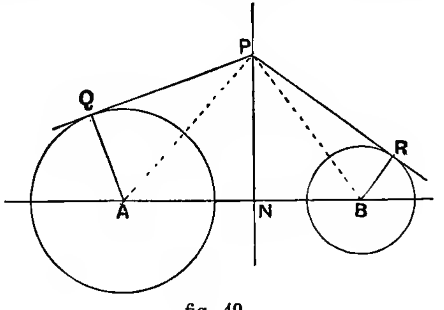

fig. 49.

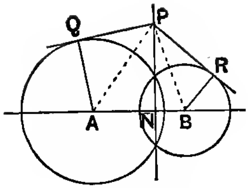

fig. 50.

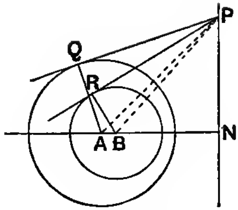

fig. 51.

[*See also fig.* 49 *on page* 87.]

A, B are the centres of the two circles.

Let P be any point on their radical axis.

Draw PQ, PR tangents to the circles, and draw PN $\perp$ to AB.

Since P is on the radical axis

$$PQ^2 = PR^2,$$

$$\therefore \; AP^2 - AQ^2 = BP^2 - BR^2,$$

$$\therefore \; PN^2 + AN^2 - AQ^2 = PN^2 + BN^2 - BR^2,$$

$$\therefore \; AN^2 - BN^2 = AQ^2 - BR^2.$$

$\therefore$ having regard to sense

$$(AN + NB)(AN - NB) = AQ^2 - BR^2,$$

$$\therefore \; AB\,(AN - NB) = AQ^2 - BR^2,$$

$\therefore$ AN $-$ NB is independent of the position of P,

$\therefore$ N is a fixed point.

But PN is $\perp$ to AB.

$\therefore$ the locus of P is a fixed straight line $\perp$ to AB cutting AB at a point N, such that $AN^2 - BN^2 = AQ^2 - BR^2$.

If we forget this last relation, it is at once recovered from the fact that the tangents from N are equal.

Note that in the case of intersecting circles the common chord is not, according to the above definition, part of the radical axis. The following exercise suggests a modification of the definition which would enable us to remove this limitation, and **regard the whole line as the radical axis.**

Ex. **381.** Prove that if from P, any point on the radical axis of two circles, lines are drawn cutting the one circle in W, X and the other in Y, Z, then PW . PX = PY . PZ. Take care that your proof applies to the common chord of two intersecting circles.

Ex. **382.** In the case of each of the following pairs of circles, find the ratio in which their radical axis cuts the line of centres. Make rough sketches of the figures. (R, r are the radii of the circles, and d the distance between their centres.)

(i) $R=5,\ r=3,\ d=10$;
(ii) $R=5,\ r=3,\ d=8$;
(iii) $R=5,\ r=3,\ d=6$;
(iv) $R=5,\ r=3,\ d=2$;
(v) $R=5,\ r=3,\ d=1$;
(vi) $R=5,\ r=3,\ d=0$;
(vii) $R=5,\ r=0,\ d=7$;
(viii) $R=5,\ r=0,\ d=3$;
(ix) $R=5,\ r=0,\ d=0$;
(x) $R=0,\ r=0,\ d=5$.

Ex. **383.** What is the radical axis of two circles, one of which has an infinite radius (i) when they cut one another, (ii) when they do not cut?

Ex. **384.** What is the radical axis of two circles which touch one another?

Ex. **385.** The radical axis of two circles bisects their common tangent.

Ex. **386.** Suggest a construction for the radical axis of two non-intersecting circles. (See Ex. 385.)

In what case does the construction fail?

Ex. **387.** In a triangle ABC, the radical axis of its in-circle and the ex-circle opposite A bisects BC and cuts AB and AC at points whose distances from A are each equal to $\frac{1}{2}(b+c)$.

Ex. **388.** If the radical axis of the ex-circles opposite A and B cut AB, AC in X, Y respectively, find the distances AX, AY.

Ex. **389.** Three circles pass through the same two points. Prove that, if the common tangent of any two of them is cut by the third circle, it is divided harmonically.

Ex. **390.** **Prove the validity of the following construction for the radical axis of two circles. Draw any circle to cut the one circle in Q, Q′ and the other in R, R′; produce QQ′ and RR′ to cut at P; draw PN ⊥ to the line of centres. Then PN is the radical axis.**

Ex. **391.** What is the radical axis of:

(i) a point-circle and a circle,

(ii) two point-circles,

(iii) a circle and a line (a circle of infinite radius),

(iv) a point-circle and a line,

(v) two concentric circles,

(vi) two parallel lines,

(vii) two intersecting lines?

Ex. **392.** Give a construction for the radical axis of a circle and a point analogous to the construction of Ex. 390.

Does your construction hold if the point is inside the circle?

Ex. **393.** If from any point P tangents are drawn to two circles, the difference of their squares is equal to twice the rectangle contained by the distance between the centres and the perpendicular from P on the radical axis of the circles.

[Join P to the centres of the circles, and from P draw a perpendicular to the line of centres.]

Theorem 41.

The three radical axes of three circles taken in pairs are concurrent.

The proof is left to the reader.

Ex. **394.** Prove Theorem 41.

[Consider the tangents from the point where two of the radical axes intersect.]

Definition. The point of concurrence of the three radical axes of a system of three circles is called the **radical centre** of the three circles.

Ex. **395.** If each of three circles touches the other two, the three common tangents at their points of contact are concurrent.

Ex. **396.** Circles are described with the sides of a triangle ABC as diameters, where is their radical centre?

[What are their radical axes?]

Ex. **397.** Where is the radical centre of three point-circles?

Ex. **398.** If the centres of three circles are collinear, where is their radical centre?

Ex. **399.** Where is the radical centre of three circles, two of which are concentric?

Ex. **399 a.** Three given circles have in general one common orthogonal circle. Discuss the exceptional cases, e.g. zero or infinite radii.

Coaxal Circles.

Ex. **400.** Draw a circle A and a circle B to touch it; what is their radical axis? Find another circle C such that A and C have the same radical axis as A and B.

Ex. **401.** Draw two intersecting circles A and B. What is their radical axis? Draw another circle C such that A and C have the same radical axis as A and B. What is the radical axis of B and C?

Ex. **402.** Draw a circle with centre A and a line PN outside it; draw AN $\perp$ to PN; from P draw PT a tangent to the circle; from P draw a line PT′ = PT, draw a circle with its centre on AN (or AN produced) to touch PT′ at T′. What is the radical axis of the two circles?

Definition. If a system of circles is such that every pair has the same radical axis, the circles are said to be **coaxal.**

It is obvious that coaxal circles have all their centres on a straight line, which is perpendicular to the common radical axis.

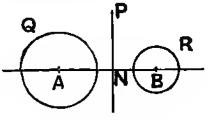

fig. 52.

In Theorem 40 it was proved that if A, B are the centres of two circles whose radii are AQ, BR and N the point at which their radical axis cuts AB, then

$$AN^2 - AQ^2 = BN^2 - BR^2.$$

By reversing the steps of that theorem we could prove that, if the given relation is true and if tangents are drawn to the circles from any point P on the perpendicular to AB through N, these tangents must be equal; in fact, that if the relation holds PN is the radical axis of the two circles.

Now suppose that the one circle (centre A) and the radical axis are given; by taking different positions for B on the line AN (produced both ways) and choosing in each instance the radius given by the above relation, we get an infinite number of circles, the tangents to which from any point P on PN are equal to one another.

We therefore get a system of coaxal circles.

Intersecting coaxal circles. If any circle of a coaxal system cuts the radical axis at C and D say, all the circles must pass through C and D, for the tangent to the one circle from C (or D) is of zero length, and therefore the tangent from C (or D) to each circle of the system must be of zero length.

In the same way, if any two circles of the system intersect at C and D, all the circles must pass through C and D, and CD is their common radical axis.

This suggests an easy construction for a system of intersecting coaxal circles.

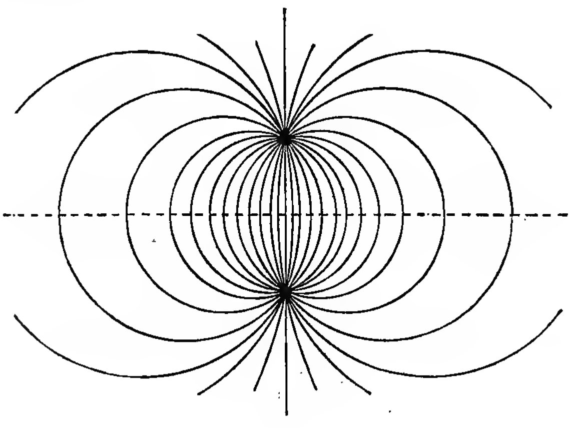

fig. 53

Non-intersecting coaxal circles. We will now consider a construction for a system of coaxal circles for the case in which no circle of the system cuts the radical axis (and no two circles of the system cut one another).

[*See fig.* 55 *on page* 94.]

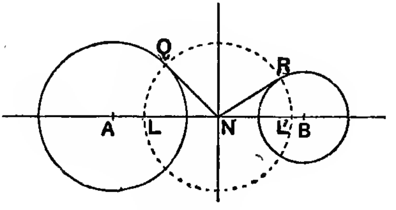

fig. 54.

Suppose we have the radical axis and one circle of the system.

From N (which must be outside all the circles) draw a tangent NQ to the circle.

With centre N and radius NQ describe a circle.

Draw BR a tangent to this circle from any suitable point in AN (or that line produced). Then the circle with B as centre and BR as radius will be a circle of the system.

For $$AN^2 - AQ^2 = NQ^2 = NR^2 = BN^2 - BR^2.$$

It should be noticed that instead of taking N as centre we might take any point on the radical axis. This method would then apply to intersecting circles as well as non-intersecting.

Ex. **403.** Draw a system of coaxal circles, one circle of the system having its centre 4 cm. from the radical axis and having a radius of 3 cm.

It is worthy of special notice that in a system of coaxal circles one member of the system consists of the radical axis and the line at infinity.

Ex. **404.** In fig. 54 what position of R will give the radical axis as a member of the system?

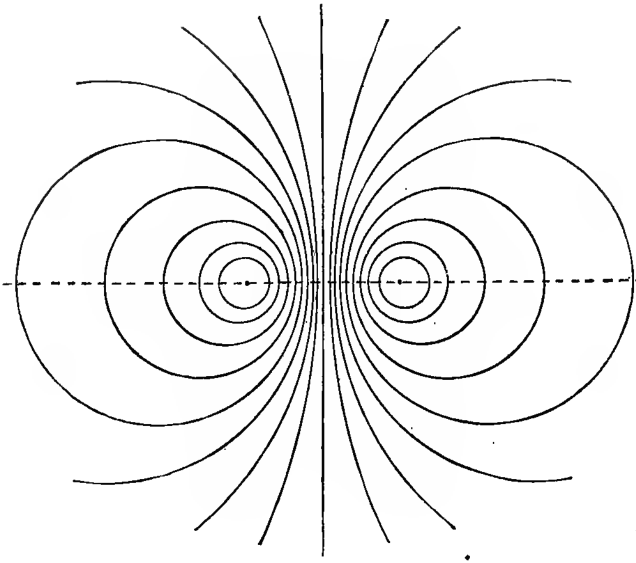

fig. 55.

Ex. **405.** From what points of the line AB in fig. 54 is it impossible to draw tangents to the construction circle?

Take a point B′ between L and N; according to the formula, what would be the square on the radius of the circle of the system with centre B′? Is this positive or negative?

Ex. **406.** What is the radius of the circle of ths system whose centre is at L, where the construction circle cuts AN?

Limiting Points. It is obvious from the method of constructing non-intersecting coaxal circles (and also from the relation $AN^2 - AQ^2 = BN^2 - BR^2$) that B cannot be within the construction circle, but may be anywhere else along the line through A and N.

The circles of the system whose centres are at the points L and L′ where the construction circle cuts the line AN have zero radius, i.e. are point circles. L and L′ are called the **limiting points** of the system.

Definition. The **limiting points** of a system of coaxal circles are the point circles of the system.

A system of intersecting coaxal circles has no real limiting points; for any point in the line of centres may be taken as the centre of a circle of the system.

Or, looking at the question from the point of view of the definition, in the case of intersecting circles there are no real point circles of the system, for $BN^2 - BR^2 = AN^2 - AQ^2$ which is negative. $\therefore$ BR^2 cannot be zero.

Ex. **407.** P is any point on the radical axis of a coaxal system of circles; with P as centre a circle is described to cut one of the circles orthogonally; what is its radius? Prove that this circle cuts all the circles of the system orthogonally.

Ex. **408.** In Ex. 407 suppose the system to be of the non-intersecting type; prove that the orthogonal circle passes through two points which are the same whatever position on the radical axis is chosen for P.

Ex. **409.** In Ex. 407 suppose the system to be of the non-intersecting type; prove that an infinite number of circles can be drawn to cut all the circles orthogonally, and prove that these cutting circles are themselves coaxal.

Theorem 42.

With every system of coaxal circles there is associated another system of coaxal circles, and each circle of either system cuts every circle of the other system orthogonally.

Since the tangents to a system of coaxal circles (A) from any point P on their radical axis are equal to one another, it follows that the circle (B) with centre P and any one of these tangents as radius will cut all the circles of the system (A) orthogonally.

Again, since there is an infinite number of positions of P on the radical axis, there is an infinite number of circles (B) each of which cuts all the circles of the system (A) orthogonally.

We have still to show that these cutting circles (B) form another coaxal system.

Consider any one circle of the system (A); the tangents from its centre to the orthogonal circles (B) are each a radius of the (A) circle, and therefore equal to one another; similarly for any other circle of the system (A).

∴ the orthogonal circles (B) are coaxal, their radical axis being the line of centres of the system (A).

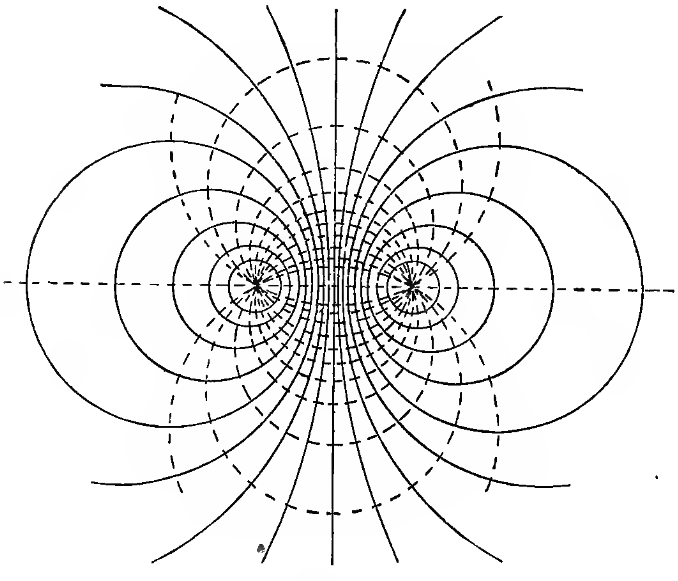

fig. 56.

THEOREM 43.

Of two orthogonal systems of coaxal circles, one system is of the intersecting type and the other of the non-intersecting type, and the limiting points of the latter are the common points of the former.

Suppose that a system (A) of coaxal circles is of the non-intersecting type and has limiting points L and L′; since L and L′ are point circles of the system, it follows that all the circles of the orthogonal system (B) pass through L and L′, and therefore that the system (B) is of the intersecting type, L and L′ being the common points.

Again suppose the given system is of the intersecting type, M and M′ being the common points; we see that no circle of the orthogonal system can have its centre between M and M′; therefore these are the limiting points of the orthogonal system.

Ex. 410. Draw a system of coaxal circles which touch one another; draw the orthogonal system. Where are their limiting points and common points?

Ex. **411.** The radical axes of a given circle and the circles of a coaxal system are concurrent.

Ex. **412.** Prove that a common tangent to any two circles is divided harmonically by any coaxal circle which cuts it.

Ex. **413.** If L is one of the limiting points of a system of coaxal circles and XLY is any chord of a circle of the system, the distances of X, L, Y from the radical axis are in geometrical progression.

Ex. **414.** A common tangent to any two circles of a non-intersecting coaxal system subtends a right angle at each of the limiting points.

Ex. **415. The polar of either limiting point of a coaxal system with regard to any circle of the system passes through the other limiting point.**

Ex. **416.** The tangents to a family of circles from a point A are all equal to one another; and the tangents from another point B are also equal to one another; prove that the circles are all coaxal. What is the condition that the system should be of the non-intersecting type, and what are the limiting points in that case?

Ex. **417.** Prove that the polars of a fixed point P with regard to a system of coaxal circles with real limiting points all pass through another fixed point, namely that point on the circle through P and the limiting points which is at the other extremity of the diameter through P.

Exercises on Chapter X.

Ex. **418.** If T be a point from which equal tangents can be drawn to two circles whose centres are A and B, prove that the chords of contact of tangents from T intersect on the line through T at right angles to AB.

Ex. **419.** The mid-points of the four common tangents to two non-intersecting circles are collinear.

Ex. **420.** If each of three circles intersects the other two, prove that their common chords are concurrent.

Ex. **421.** Three circles, centres D, E, F, touch each other two and two in A, B, C. Prove that the circumcircle of ABC is the in-circle of DEF.

Ex. **422.** Show how to describe a circle of a given coaxal system to touch another given circle (i) when the system is of the intersecting, (ii) of the non-intersecting type.

Ex. **423.** Consider the various Apollonius' circles for two fixed points obtained by varying the given ratio; are they coaxal?

Ex. **424.** If a system of circles have the same polar with regard to a given point, show that they are coaxal, and find the position of the common radical axis.

Ex. **425.** Prove that the four circles whose diameters are the common tangents to two non-intersecting circles have a common radical axis.

Ex. **426.** Show that the limiting points of a pair of non-intersecting circles and the points of contact of any one of their common tangents lie on a circle cutting the two circles orthogonally.

Ex. **427.** The circle whose diameter is the line joining the centres of similitude of two circles is coaxal with those circles.

Ex. **428.** If two circles X and Y cut orthogonally, prove that the polar with respect to X of any point A on Y passes through B, the point diametrically opposite to A.

If the polars of a point, with respect to three circles whose centres are on a straight line, are concurrent, prove that the three circles are coaxal.

Ex. 429. Prove that the common orthogonal circle of three given circles is the locus of a point whose polars with respect to the three circles are concurrent.

Ex. **430.** The external common tangent to two circles which lie outside one another touches them in A and B; show that the circle described on AB as diameter passes through the limiting points L and L′ of the coaxal system to which the circles belong.

If O is the mid-point of the above common tangent, prove that OL, OL′ are parallel to the internal common tangents of the circles.

Ex. **431.** The internal and external bisectors of the angles of a triangle are drawn, and on the lengths they intercept on the opposite sides circles are described having these intercepts as diameters; prove that these circles all pass through two points which are collinear with the circumcentre of the triangle.

Ex. **432.** Describe a circle which shall pass through two given points and bisect the circumference of a given circle.

Ex. **433.** Prove that all the circles which bisect the circumferences of two given circles pass through two common points.

Ex. **434.** ABC is a triangle and two circles are drawn, one to tonch AB at A and to pass through C, the other to touch AC at A and to pass through B. If the common chord of these circles meets BC in D, prove that $BD : DC = BA^2 : AC^2$.

Ex. **435.** A line PQ is drawn touching at P a circle of a coaxal system of which the limiting points are K, K′, and Q is a point on the line on the opposite side of the radical axis to P; show that, if T, T′ be the lengths of the tangents drawn from P to the two concentric circles of which the common centre is Q and radii are respectively QK, QK′, then

$$T : T' :: PK : PK'.$$

Ex. **436.** The tangents at A, A′ to one given circle cut a given non-intersecting circle in P, Q and P′, Q′ respectively, and AA′ cuts PP′ in X. Show that, if O is a limiting point of the coaxal system determined by the two given circles, then will OX be a bisector of the angle POP′.

CHAPTER XI.

INVERSION.

Definition. If O is a fixed point and P any other point, and if a point P′ is taken in OP (produced if necessary) such that OP . OP′ $= k^2$ (where k is a constant), P′ is said to be the **inverse** of P with regard to the circle whose centre is O and radius k. O is called the **centre of inversion**, the circle is called the **circle of inversion**, and k the **radius of inversion***.

From the definition it is obvious that P is the inverse of P′. Also that OP′ varies inversely as OP; hence the name.

If P is made to describe any given figure and if P′ always moves so that it is the inverse of P, P′ describes a figure which is called the inverse of the given figure with respect to the circle of inversion.

When a large number of inverse points have to be found the following construction is useful.

Assume the above notation; draw a circle of any radius and a tangent to it at any point A; from the tangent cut off a length Ao = the radius of inversion; find a point p on the circle such that op = OP; let p' be the other point at which op cuts the circle, then op' = OP′.

Ex. **437.** What is the inverse of a straight line when the centre of inversion is on the straight line?

Ex. **438.** What is the inverse of a given circle when the centre of inversion is the centre of the given circle?

* Sometimes k^2 is called the **constant of inversion.**

Ex. **439.** Draw a straight line and mark a point O 4 inches from the line; taking O as centre and a radius of inversion 3 inches, mark a number of points on the inverse of the straight line.

Ex. **440.** Draw a circle of radius 2 inches; take a point O 1 inch from its centre; taking O as centre and 1 inch as radius of inversion, plot the inverse of the circle.

Ex. **441.** Draw a circle of radius 2 inches; take a point O 3 inches from its centre; taking O as centre and 2 inches as radius of inversion, plot the inverse of the circle.

Ex. **442.** Plot a parabola and invert it (i) with the focus as centre of inversion, (ii) with the vertex as centre of inversion.

Theorem 44.

If a figure is inverted first with one radius of inversion and then with a different radius, the centre being the same in both cases, the two inverse figures are similar and similarly situated, the centre being their centre of similitude.

If P_1 is the inverse of a point P when k_1 is the radius of inversion and P_2 the inverse of the same point when k_2 is the radius, the centre O being the same in both cases, then

$$OP_1 : OP_2 = k_1^2 : k_2^2.$$

Hence the theorem is true.

In consequence of this property it is generally unnecessary to specify the radius of inversion; in fact, it is usual to make no reference to the circle of inversion and to speak of inverting with regard to a point.

Sometimes we take a negative constant of inversion; in this case the circle of inversion must of necessity be avoided as it has an imaginary radius.

Theorem 45.

The inverse of a straight line, with regard to a point on it, is the line itself.

This is obvious from the definition.

Ex. **443.** What is the inverse of a point on the line which is infinitely close to the centre of inversion?

Ex. **444.** What is the inverse of the line at infinity?

Ex. **445.** OABC is a straight line, and A′, B′, C′ are the inverses of A, B, C, when O is the centre of inversion; if B is the mid-point of AC, prove that O, A′, B′, C′ is a harmonic range.

Ex. **446.** If a harmonic range is inverted with regard to any point on the line, another harmonic range is obtained.

Ex. **447.** Prove that Ex. 445 is a particular case of Ex. 446.

Theorem 46.

The inverse of a straight line, with regard to a point outside it, is a circle through the centre of inversion.

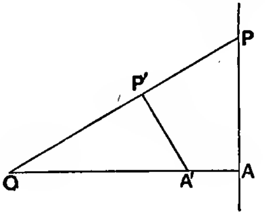

fig. 57.

Let PA be the given line and O be the centre of inversion.

Draw OA ⊥ to PA.

Take A′, P′ the inverses of A, P.

Then $OP \cdot OP' = OA \cdot OA'$,

∴ P, A, A′, P′ are concyclic,

∴ $\angle OP'A' = \angle OAP$

$= \text{a rt.} \angle.$

But O and A′ are fixed points.

∴ as P moves along the line PA, P′ describes a circle on OA′ as diameter.

Ex. **448.** Show that Theorem 45 is not really an exception to the theorem that the inverse of a straight line is a circle through the centre of inversion.

Ex. **449.** Draw a figure showing the inverse of a straight line with regard to a point outside it for a negative constant of inversion.

Theorem 47.

The inverse of a circle with regard to a point on its circumference is a straight line at right angles to the diameter through the centre of inversion.

The proof is left to the reader.

Ex. **450.** Prove Theorem 47.

Ex. **451.** **If a circle is inverted with regard to a point on it, the centre of the circle inverts into the image of the centre of inversion in the resulting straight line.**

Ex. **452.** A straight line meets a circle α in the points A, B and a circle β in the points C, D; O is a point on the radical axis of α and β. OA, OB meet α again at A′, B′ and OC, OD meet β again at C′, D′. Show that O, A′, B′, C′, D′ lie on a circle.

Ex. **453.** In fig. 58, $OA = OB$, $AP = PB = BP' = P'A$;

(i) prove that OPP' is a straight line;

(ii) prove that, if O be fixed, P and P′ will move so that they are inverse points with regard to O.

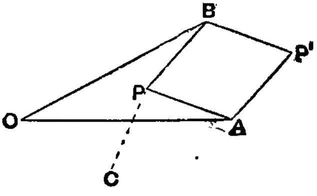

fig. 58.

Ex. **454.** If, in Ex. 453, C is a fixed point, and P moves so that $CP = CO$, prove that the locus of P′ is a straight line.

Peaucellier's Cell*. Fig. 58 suggests a mechanical device, called a linkage, for constructing the inverse of a given figure; a model can be constructed consisting of rods freely hinged at the points O, A, B, P, P′; from Ex. 453 we see that if O is fixed and P moved along a given curve P′ describes the inverse curve.

Ex. 454 shows that, if P is made to describe a circle through O, P′ moves on a straight line. Now it is not necessary that the rods should be straight; the only essential is that the distance between the points O and A should equal that between O and B, etc., and the equality of these distances can be tested by superposition. Thus this linkage enables us to draw a straight line without presupposing that we have a straight edge.

* This linkage was invented in 1873 by Peaucellier, a captain in the French army.

Theorem 48.

The inverse of a circle with regard to a point not on its circumference is another circle.

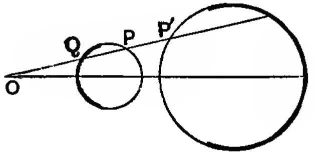

fig. 59.

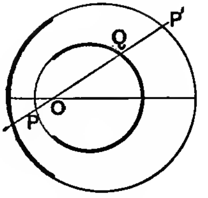

fig. 60.

Let O be the centre of inversion.

Draw a line OPQ to cut the circle at P and Q.

Let P′ be the inverse of P.

$$\text{Then } OP \cdot OP' = \text{constant.}$$

$$\text{But } OP \cdot OQ = \text{constant,}$$

$$\therefore\ OP' : OQ = \text{constant.}$$

But the locus of Q is a circle,

∴ the locus of P′ is a circle (Chap. VIII. § 1).

I.e. the inverse of the given circle is a circle.

Note that in figs. 59 and 60 the parts of the circles which are thickened are inverses of one another.

Ex. **455.** Show how to invert a circle into itself, the centre of inversion being outside the circle.

Ex. **456.** Is it possible to invert a circle into itself (i) with regard to a point inside the circle, (ii) with regard to a point on the circle?

Ex. **457.** Show how to invert simultaneously each of three circles into itself.

Ex. **458.** **If a circle is inverted with regard to any point not on its circumference, its centre inverts into the point at which the line of centres cuts the polar of the centre of inversion with respect to the inverse circle.**

Ex. **459.** Show that Ex. 451 is a particular case of Ex. 458.

Theorem 49.

Two curves intersect at the same angles as their inverses.

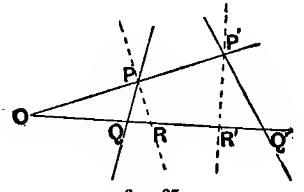

fig. 61.

Let P be a point of intersection.

Through O, the centre of inversion, draw a straight line making a small angle with OP to cut the curves in Q and R.

Let P′, Q′, R′ be the inverses of P, Q, R respectively.

Join PQ, PR, P′R′, P′Q′.

Since $OP \cdot OP' = OQ \cdot OQ'$,

$\therefore$ P, P′, Q′, Q are concyclic,

$\therefore \angle OPQ = \angle OQ'P'$.

Similarly $\angle OPR = \angle OR'P'$,

$\therefore \angle QPR = \angle R'P'Q'$.

Now, as OQ moves up to OP, so PQ, PR, P′Q′, P′R′ move up to and in the limit coincide with the tangents to the curves at P and P′.

$\therefore$ the angles between the tangents at P are equal to the angles between the tangents at P′.

$\therefore$ two curves cut at the same angles as their inverses.

Ex. **460.** Give an independent proof of Theorem 49 in the case of two straight lines inverted into a straight line and a circle.

Ex. **461.** Give an independent proof of Theorem 49 in the case of two straight lines inverted into two circles.

Ex. **462.** Prove that the tangent to a curve from the centre of inversion is also a tangent to the inverse curve.

By applying the above results we can deduce new theorems from theorems we know already; this process is called inverting a theorem.

EXAMPLE I.

Invert the following theorem with regard to the point O :—

If O, A, B, C are four points on a circle, angles AOC, ABC are equal or supplementary.

Let A′, B′, C′ be the inverses of A, B, C.

We will write the corresponding properties of the figure and its inverse in parallel columns.

[It is convenient to draw the figures separately.]

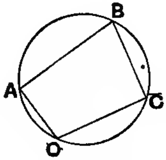

fig. 62.

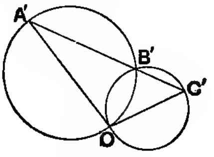

fig. 63.

OABC is a circle,	A′B′C′ is a straight line,
OA, OC are st. lines,	OA′, OC′ are st. lines,
AB is a st. line,	OA′B′ is a circle,
BC is a st. line,	OB′C′ is a circle,
∠ AOC is equal or supplementary to ∠ ABC.	∠ A′OC′ is equal or supplementary to ∠ at which circles OA′B′, OB′C′ intersect.

Hence we deduce the theorem that, if A′B′C′ is a straight line, and O a point outside it, the angles at which the circles OB′A′, OB′C′ intersect are equal or supplementary to the angle A′OC′.

Example II.

Prove the following theorem by inverting with regard to the point O. AOBF, AOCE are two circles intersecting at O, A; FO a diameter of the first cuts the second at C, EO a diameter of the second cuts the first at B; then AO passes through the centre of the circle OBC.

Let A′, B′, ... be the inverses of A, B, ...

We will now write the corresponding properties of the figure and its inverse in parallel columns.

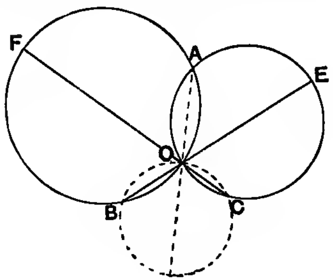

fig. 64.

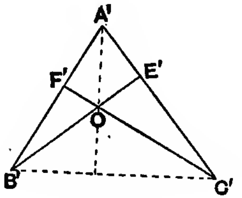

fig. 65.

AOBF, AOCE are two circles through A, O,	A′B′F′, A′C′E′ are two st. lines through A′,
FO, a diameter of ⊙AOBF, cuts ⊙AOCE at C;	F′O, the perpendicular from O on A′B′F′, cuts A′C′E′ at C′;
EO, a diameter of ⊙AOCE, cuts ⊙AOBF at B.	E′O, the perpendicular from O on A′C′E′, cuts A′B′F′ at B′.
To prove that AO passes through the centre of ⊙OBC.	To prove that AO is perpendicular to B′C′.

Now we see that the inverse theorem is true (it is the orthocentre property of a triangle).

∴ the original theorem is true.

Ex. **463.** Invert the following theorem with regard to the point O: If O, A, B, C are four points on a circle, angles OAC, OBC are equal or supplementary.

Ex. **464.** Invert the theorem 'The angle in a semicircle is a right angle' with regard to one end of the diameter.

Ex. **465.** OP and OQ are lines through a fixed point O, inclined at a constant angle and intersecting a fixed line in P, Q; the envelope of the circle round OPQ will be another circle.

Ex. **466.** Prove by inversion (or otherwise) that if the circumcircles of two triangles ABC, ABD cut orthogonally, then the circumcircles of CAD and CBD also cut orthogonally.

Ex. **467.** Prove by inversion that the circles having for diameters three chords OA, OB, OC of a circle intersect again by pairs in three collinear points.

Ex. **468.** Three circles OBC, OBE, OCF pass through a point O; OBF is a straight line passing through the centre of the circle OCF; OCE is a straight line passing through the centre of the circle OBE; prove that circles OBE, OCF intersect on OD the diameter through O of the circle OBC.

Ex. **469.** Prove by inversion that a straight line drawn through a point O to cut a circle is divided harmonically by the circle and the polar of O.

[Invert with regard to O.]

Ex. **470. The limiting points of a coaxal system are inverse points with regard to any circle of the system.**

Ex. **471. A system of intersecting coaxal circles inverted with regard to a point of intersection becomes a system of straight lines through a point.**

Ex. **472.** Invert the following theorem with regard to the point O: If each of a system of circles passes through two given points O and O′, another system of circles can be described which cut the circles of the first system orthogonally.

Ex. **473. A system of non-intersecting coaxal circles inverted with respect to a limiting point of the system becomes a system of concentric circles having the inverse of the other limiting point for centre.**

[Consider the orthogonal system of circles and use Ex. 472.]

Ex. **474.** What is the inverse of a system of intersecting coaxal circles with respect to any point?

Ex. **475.** What is the inverse of a system of non-intersecting coaxal circles with respect to any point?

Inversion may be applied to geometry of three dimensions.

By rotating the figures of theorems 46, 47, 48 about the line through the centre of inversion and the centres of the circles we arrive at the following results:

(i) The inverse of a plane with regard to a point outside it is a sphere through the centre of inversion.

(ii) The inverse of a sphere with regard to a point on its surface is a plane at right angles to the diameter through the centre of inversion.

(iii) The inverse of a sphere with regard to a point not on its surface is another sphere.

Ex. **476.** What is the inverse of a circle with regard to a point not in its plane?

[Regard the circle as the intersection of a sphere and a plane.]

Ex. **477.** A circle is inverted with respect to a sphere whose centre O does not lie in the plane of the circle; prove that the inverse is a circle, and show that the point P which inverts into the centre of the inverse circle is obtained thus: Describe a sphere through O and the circumference of the given circle; join O to the pole of the plane of the circle with respect to this sphere; this line cuts the sphere at P.

Exercises on Chapter XI.

Ex. **478.** Q is the inverse of P with respect to a circle whose centre is O, AQB is any chord of the circle; prove that PQ bisects the angle APB.

Ex. **479.** A circle, its inverse, and the circle of inversion are coaxal with one another.

Ex. **480.** Show that it is possible to invert three circles so that the centres of the inverse circles are collinear.

Ex. **481.** If two circles cut orthogonally the inverse of the centre of the first with respect to the second coincides with the inverse of the centre of the second with respect to the first.

Ex. **482.** Two points are inverse with respect to a circle; show that, if the figure be inverted with respect to any circle, the new figure will have the same property.

Ex. **483.** A, B, C are three points in a straight line and P any other point. AFE, BFD, CED are drawn perpendicular to PA, PB, PC respectively; prove that P, D, E, F lie in a circle.

Obtain a new theorem by inverting with respect to P.

Ex. **484.** If c is the distance between the centres of two intersecting circles whose radii are r, r', show that the ratio $c^2-r^2-r'^2 : rr'$ is unaltered by inversion with regard to any point external to the two circles.

Ex. **485.** Two circles intersect at O and P and their tangents at O meet the circles again at A and B. Show that the circle circumscribing the triangle AOB cuts OP produced at a point Q such that OQ$=$2OP, and that if a line is drawn through P parallel to the tangent at O to the circle AOB, then the part of this line intercepted between OA and OB is bisected at P.

Ex. **486.** If A, B, C be three collinear points and O any other point, show that the centres P, Q, R of the three circles circumscribing the triangles OBC, OCA, OAB are concyclic with O. Also that if three other circles are drawn through O, A; O, B; O, C to cut the circles OBC, OCA, OAB, respectively, at right angles, then these three circles will meet in a point which lies on the circumcircle of the quadrilateral OPQR.

Ex. **487.** From any point P on the circle ABC a pair of tangents PQ, PR are drawn to the circle DEF and the chord QR is bisected in S. Show that the locus of S is a circle; except when the circle ABC passes through the centre of the circle DEF, in which case the locus of S is a straight line.

Ex. **488.** Through one of the points of intersection of two given circles any line is drawn which cuts the circles again in P, Q respectively. Prove that the middle point of PQ is on a circle whose centre is midway between the centres of the given circles.

Ex. **489.** Show that there is in general one circle of a coaxal system which cuts a given circle orthogonally.

What is the exceptional case?

Ex. **490.** Show that circles which cut one given circle orthogonally and another given circle at a given angle will also cut a third fixed circle at the same fixed angle.

Ex. **491.** A, B, C, D are four coplanar points. Prove that in an infinite number of ways two circles can be drawn making an assigned angle with each other, and such that A and B are a pair of inverse points of one circle, and C and D of the other circle.

Ex. **492.** If P', Q' are the inverses of P, Q with respect to a point O, $PQ : P'Q' = OP . OQ : k^2$, where k^2 is the constant of inversion.

Ex. **493.** Invert with respect to the point O the proposition: If PAQ, RAS are two chords of a circle which passes through O, the rectangle PA . AQ = rectangle RA . AS.

Ex. **494.** The sides of a triangle ABC touch a circle whose centre is O, and on OB, OC produced, if necessary, are taken points B' and C' respectively such that $OB . OB' = OC . OC' = OA^2$. Prove that O is the orthocentre of the triangle $AB'C'$.

Ex. **495.** Two given circles intersect in a point O; prove, by the method of inversion, that the inverse point of O with respect to any circle which touches them lies on one or other of two fixed circles which cut one another orthogonally.

Ex. **496.** If two circles be inverted with respect to a circle whose centre is at their external centre of similitude and whose $(\text{radius})^2$ is equal to the rectangle contained by the tangents to the circles from its centre, prove that the radical axis of the two circles inverts into the circle on the line joining the two centres of similitude as diameter.

Ex. **497.** Prove that any two circles are inverse to one another with respect to some third circle; and that with any point on this third circle as origin of inversion the two circles will invert into equal circles.

Ex. **498.** (i) A sphere is inverted from a point on its surface; show that to a system of parallels and meridians on the surface will correspond two systems of coaxal circles in the inverse figure.

(ii) Prove that, if P, Q be the ends of a diameter of a small circle of a sphere, O a point of the great circle PQ, and R any point on the circle, then the arcs of the small circles PRO, RQO are perpendicular to each other at R.

Ex. **499.** (i) A circle is inverted from a point which is not upon its circumference and not necessarily in the plane of the circle. Show that the inverse curve is also a circle.

(ii) Circles are drawn to cut a given circle orthogonally at two points of intersection and to pass through a given point not in the plane of the circle. Show that they intersect in another common point; and hence show how a circle and a point not in its plane may be inverted respectively into circle and centre.

Ex. **500.** Show that the locus of points with respect to which an anchor ring can be inverted into another anchor ring consists of a straight line and a circle.

Ex. **501.** The figures inverse to a given figure with regard to two circles C_1 and C_2 are denoted by S_1 and S_2 respectively; show that, if C_1 and C_2 cut orthogonally, the inverse of S_1 with regard to C_2 is also the inverse of S_2 with regard to C_1.

Ex. **502.** Γ is a circle and P and Q are any two points inverse to it; Γ', P', Q' are the respective inverses with regard to any point. Show that P', Q' are inverse points with regard to the circle Γ'.

Ex. **503.** (i) Show that, if the circles inverse to two given circles ACD, BCD with respect to a point P be equal, the circle PCD bisects (internally or externally) the angles of intersection of the two given circles.

(ii) Prove that four points P, Q, R, S can be found such that with respect to any one of them the points inverse to four given points A, B, C, D form a triangle and its orthocentre; and that the points inverse to P, Q, R, S with respect to any one of the four A, B, C, D also form a triangle and its orthocentre.

Ex. **504.** A circle moving in a plane always touches a fixed circle, and the tangent to the moving circle from a fixed point is always of constant length. Prove that the moving circle always touches another fixed circle.

CHAPTER XII.

ORTHOGONAL PROJECTION.

Suppose that we have a plane (say a sheet of glass) with a variety of figures drawn upon it.

And let this plane be placed, in an inclined position, above a second—horizontal—plane.

If a distant light (e.g. the sun) be allowed to shine upon the figures drawn on the glass, and to cast shadows of them upon the horizontal plane, these shadows would be 'projections' of the original figures.

If the sun is directly overhead, so that its rays strike perpendicularly upon the horizontal plane, the projection is called 'orthogonal.'

The definition of orthogonal projection is as follows.

Definition. Let there be an assembly of points (a) in a plane (p). From each point let a perpendicular be drawn to a second plane (q). The feet of these perpendiculars together constitute the orthogonal projection of the assembly (a).

We must now enquire what relations exist between figures and their orthogonal projections upon other 'planes of projection.'

In what follows, it must be assumed that the projection is orthogonal unless the contrary is stated or distinctly implied.

1. **The projection of a straight line is a straight line.**

The perpendiculars from all points on the original line form a plane, which cuts the plane of projection in a straight lino.

2. **A point of intersection of two curves in the original plane projects into a point of intersection of the resulting curves.**

3. **A tangent to a curve, and its point of contact, project into a tangent to the resulting curve and its point of contact.**

4. The lengths of lines are usually altered by orthogonal projection; in fact, the lines are foreshortened.

Ex. **505.** Take the case of projection on to a horizontal plane from a plane inclined to it at 60°.

Prove that all the lines of steepest slope are halved by projection.

Are any lines unaltered by projection?

What is the condition that two lines that are equal before projection shall remain equal after projection?

If a be the length of a segment of one of the lines of steepest slope in a plane, and θ the angle which the plane makes with the plane of projection, then $a \cos \theta$ is the length of the projection of a.

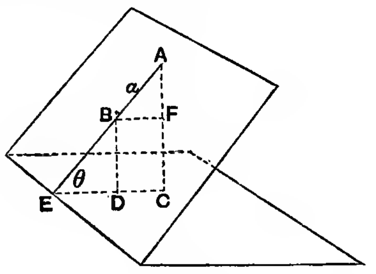

fig. 66.

AB is the segment a, CD is its projection.

In the plane AEC draw BF $\parallel$ to DC, meeting AC in F.

$$\text{Then } \angle \mathrm{ABF} = \angle \mathrm{AEC} = \theta,$$

$$\therefore \mathrm{DC} = \mathrm{BF} = a \cos \theta.$$

5. **Lines parallel to the plane of projection are unaltered in length by projection.**

6. **If A be an area in a plane, its projection has area A $\cos\theta$.**

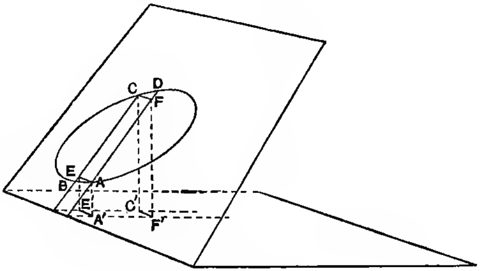

fig. 67.

Let the area be divided up into strips ABCD by lines of steepest slope.

By drawing parallels to the plane of projection, cut off a rectangle AECF from each strip.

Let A'E'C'F' be the projection of AECF.

Now $A'F' = AF \cos\theta$, $A'E' = AE$.

$\therefore$ rect. $A'E'C'F' =$ rect. $AECF \times \cos\theta$.

If the strips become very narrow (and therefore numerous), then each strip tends to equality with the corresponding rectangle, the neglected portions being comparatively unimportant; and it is shown in the infinitesimal calculus that, in the limit, no error is made by regarding the area as composed of infinitely narrow rectangles.

But each rectangle is diminished by projection, in the ratio $\cos\theta : 1$.

$\therefore$ the projection has area A $\cos\theta$.

Ex. **506.** Give an independent proof of the above theorem for a triangle, by drawing through its vertices perpendiculars to the line of intersection of the planes, and considering the three trapezia thus formed.

Hence prove the theorem for any rectilinear figure.

7. Parallel lines project into parallel lines.

The intersection of the two parallel lines is a point at infinity.

This projects into a point at infinity.

Therefore the two projected lines are parallel.

8. Parallel lines are diminished, by projection, in the same ratio.

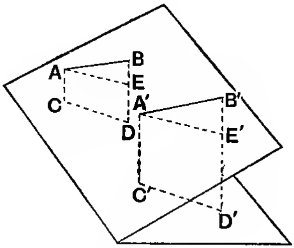

fig. 68.

AB, A′B′ are parallel; CD, C′D′ are their projections.

Draw AE ‖ to CD, A′E′ to C′D′.

Let AE meet BD in E, A′E′ meet B′D′ in E′.

Now CD is ‖ to C′D′ by (7). Thus we have AE ‖ to CD, CD ‖ to C′D′, C′D′ ‖ to A′E′.

$$\therefore \text{ AE is } \parallel \text{ to A}'\text{E}'.$$

Also AB is ‖ to A′B′.

∴ by a theorem in solid geometry

$$\angle \text{BAE} = \angle \text{B}'\text{A}'\text{E}', \ = \phi \text{ (say)},$$

$$\therefore \text{ AE} = \text{AB} \cos\phi, \ \text{A}'\text{E}' = \text{A}'\text{B}' \cos\phi.$$

But AE = CD, the proj^n of AB, and A′E′ = C′D′, the proj^n of A′B′.

Therefore the two parallel lines are both diminished in the same ratio by projection.

9. **If a line and any number of points on it be projected, the projection is divided in the same ratio as the original line.**

This follows from (8). The following particular case is useful.

10. **The projection of the mid-point of a line bisects the projection of the line.**

11. It has been seen that a number of geometrical relations are unaltered by orthogonal projection; and the beginner may be tempted to apply this principle too freely.

It must be noted that, as a rule, angle properties are destroyed by orthogonal projection.

Ex. **507.** Discover cases in which a right angle is unaltered by projection.

Ex. **508.** One arm of an angle is ‖ to the plane of projection. Is the angle increased or diminished by projection?

Ex. **509.** One arm of an angle is a line of greatest slope. Is the angle increased or diminished by projection?

Ex. **510.** Answer the question of Ex. 509 for an angle whose bisector is

(i) a line of greatest slope,

(ii) a parallel to the plane of projection.

Ex. **511.** Discover any case in which the relation of an angle and its bisector is unchanged by projection.

Ex. **512.** Prove that the relation of Ex. 511 is not preserved generally, by considering the particular case of

(i) a right angle with one arm ‖ to the plane of projection,

(ii) a square and its diagonal.

Ex. **513.** Ascertain which of the following relations are unchanged by projection, (*a*) generally, (*b*) in particular cases :

(i) triangle and orthocentre,
(ii) triangle and circumcentre,
(iii) triangle and centroid,
(iv) isosceles triangle,
(v) right-angled triangle,
(vi) parallelogram,
(vii) rectangle,
(viii) rhombus,
(ix) trapezium,
(x) circle,
(xi) a set of equivalent triangles, on the same base and on the same side of it,
(xii) a set of triangles with the same base and equal vertical angles.

Ex. **514.** If the original plane is covered with squared paper, what is the corresponding pattern on the plane of projection?

Ex. **515.** If a triangle is projected orthogonally, the centroid of the triangle projects into the centroid of the projection.

The Ellipse.

The most interesting application of the method of orthogonal projection is that derived from the circle.

The circle projects into an oval curve called an **ellipse**; it is flattened or foreshortened along the lines of steepest slope, while the dimensions parallel to the plane of projection are unaltered.

If we define the ellipse, for present purposes, as the curve whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

it is easy to prove that the ellipse is the projection of a circle.

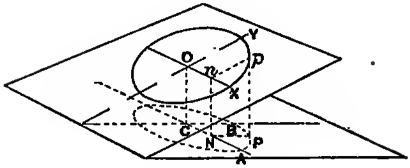

fig. 69.

Let the circle (centre O) be referred to rectangular axes OX, OY; OX being $\parallel$ to the plane of projection.

The coordinates of a point p on the $\odot$ are On, pn.

Let O$n = x$, $pn = \text{Y}$, radius $= a$.

$$\text{Then } x^2 + \text{Y}^2 = a^2.$$

The projections of OX, OY are the perpendicular lines CA, CB; these shall be the axes for the ellipse.

The coordinates of the point P on the ellipse are CN, PN.

$$\text{Now } \text{CN} = \text{O}n = x.$$

$$\text{Let } \text{PN} = y.$$

$$\text{Then } y = \text{Y} \cos\theta,$$

$$\therefore \ x^2 + \frac{y^2}{\cos^2\theta} = a^2,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{a^2\cos^2\theta} = 1.$$

But CB, the projection of OY, $= a\cos\theta$. Let CB $= b$.

Then the coordinates of P satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The locus of P is therefore an ellipse whose semiaxes are CA (a) and CB (b).

The angle properties of the circle do not admit of transference to the ellipse. But there are many important properties that may be transferred, and the chief of these are given in the following exercises.

Ex. 516. Prove the following properties of the ellipse, by first proving the allied property of the circle, and then carefully showing that the property admits of projection.

(1) Every chord of the ellipse through C is bisected at C. (These chords are called diameters.)

(2) The tangents at the extremities of a diameter are parallel.

(3) The locus of the mid-points of a series of parallel chords is a straight line, namely a diameter.

(4) If a diameter CP bisects chords parallel to a diameter CD, then CD bisects chords parallel to CP.

(Such diameters are called *conjugate.*)

(5) The lines joining a point on an ellipse to the extremities of a diameter are parallel to a pair of conjugate diameters.

(6) A diameter bisects all chords parallel to the tangents at its extremities.

(7) If a pair of conjugate diameters meet the tangent at P in T, T′, and CD be conjugate to CP, then $PT . PT' = CD^2$.

(8) The chord of contact of the tangents from T is bisected by CT.

(9) If CT meet the curve in P and the chord of contact of the tangents from T in N, then

$$CN . CT = CP^2.$$

(10) Through a point O are drawn two chords pOp', qOq'; and diameters PCP′, QCQ′ are drawn ∥ to the chords. Then

$$Op . Op' : Oq . Oq' = CP^2 : CQ^2.$$

(11) Tangents Tp, Tq are drawn from T, and PCP′, QCQ′ are the parallel diameters. Then

$$Tp^2 : Tq^2 = CP^2 : CQ^2.$$

(12) PCP', DCD' are a fixed pair of conjugate diameters; Q is a variable point on the ellipse. QV is drawn $\parallel$ to DC to meet PCP' in V. Then

$$QV^2 : PV \cdot VP' = CD^2 : CP^2 = \text{constant}.$$

(13) The area of the ellipse is πab.

(14) A circumscribing parallelogram is formed by the tangents at the extremities of a pair of conjugate diameters. Its area is constant and equal to $4ab$.

(15) CP, CD being conjugate semi-diameters,

$$CP^2 + CD^2 = \text{constant} = a^2 + b^2.$$

(16) If all the ordinates of a circle be reduced in a fixed ratio, the resulting curve is an ellipse.

Ex. **517.** By the method of projection, discover some harmonic properties of the ellipse.

Ex. **518.** From a point P on an ellipse a perpendicular PN is drawn to the major axis ACA'; NQ is drawn parallel to AP and meets CP in Q. Prove that AQ is parallel to the tangent at P.

CHAPTER XIII.

CROSS-RATIO.

Definition. A system of points on a straight line is called a **range**; the line is called the **base** of the range.

Definition. A system of lines through a point is called a **pencil**; the point is called the **vertex** of the pencil.

Definition. If A, B, C, D be a range of 4 points, and if C, D be regarded as dividing the line AB (internally or externally), then $\frac{AC}{CB} : \frac{AD}{DB}$ is called a **cross-ratio** or **anharmonic ratio** of the range ABCD, and is written $\{AB, CD\}$; the sense of lines is taken into account.

Ranges of equal cross-ratio are called **equicross**.

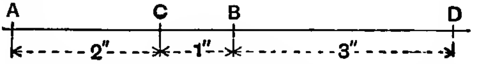

fig. 70.

Ex. **519.** Calculate $\{AB, CD\}$ for the above range. Also calculate $\{CD, AB\}$, $\{AC, BD\}$, and all the other cross-ratios obtainable by pairing the points in different ways.

• Ex. **520.** If a range ABCD is inverted, with respect to a point on the same line, into a range A′B′C′D′, then $\{AB, CD\} = \{A'B', C'D'\}$. Examine what this leads to if A coincides with O, and $\{OB, CD\}$ is harmonic.

Ex. **521.** If a pencil of four lines is cut by two parallel lines in ranges ABCD, A′B′C′D′, then $\{AB, CD\} = \{A'B', C'D'\}$.

Ex. **522.** If $\{AB, CD\} = \{AB, CE\}$, then the points D and E coincide.

Ex. **523.** The projection of a range ABCD on any line is A′B′C′D′; prove that $\{AB, CD\} = \{A'B', C'D'\}$.

Ex. **524.** Investigate the cases

$$\{AB, CD\} = 1, \ \{AB, CD\} = 0, \ \{AB, CD\} = \infty.$$

From the definition of cross-ratio, it is clear that one of the cross-ratios of a **harmonic range** is equal to -1.

Ex. **525.** **If $\{AB, CD\} = \{AB, DC\}$, then ACBD is a harmonic range.**

Ex. **526.** If A, B, C, D be collinear, and C′, D′ be the harmonic conjugates of C, D respectively with respect to A, B; then

$$\{AB, CD\} = \{AB, C'D'\}.$$

As four letters admit of twenty-four permutations, the cross-ratio of a range A, B, C, D can be written down in twenty-four ways. These will not give rise, however, to twenty-four different cross-ratios.

To begin with, $\{AB, CD\} = \{CD, AB\}$;

$$\text{for } \{AB, CD\} = \frac{AC}{CB} : \frac{AD}{DB} = \frac{AC \cdot DB}{CB \cdot AD},$$

$$\text{and } \{CD, AB\} = \frac{CA}{AD} : \frac{CB}{BD} = \frac{CA \cdot BD}{AD \cdot CB} = \frac{AC \cdot DB}{CB \cdot AD}.$$

In the same way it is shown that

$$\{AB, CD\} = \{BA, DC\}.$$

$$\text{Thus } \{AB, CD\} = \{CD, AB\} = \{BA, DC\} = \{DC, BA\},$$

a group of four equal cross-ratios.

This reduces the possible number of different cross-ratios to six; and it will now be shown that these six are generally unequal.

$$\text{For, let } \{AB, CD\} = \frac{AC \cdot DB}{CB \cdot AD} = \lambda.$$

Then, interchanging the first pair,

$$\{BA, CD\} = \frac{BC \, . \, DA}{CA \, . \, BD} = \frac{CB \, . \, AD}{AC \, . \, DB} = \frac{1}{\lambda}.$$

Again, $\{AC, BD\} = 1 - \lambda.$

For $AB \, . \, CD + AC \, . \, DB + AD \, . \, BC = 0.$ (See Ex. 2, p. 4.)

$$\therefore \frac{AB \, . \, CD}{AD \, . \, BC} + \frac{AC \, . \, DB}{AD \, . \, BC} + 1 = 0.$$

$$\text{But } \frac{AB \, . \, CD}{AD \, . \, BC} = -\frac{AB}{BC}\Big/\frac{AD}{DC} = -\{AC, BD\},$$

$$\text{and } \frac{AC \, . \, DB}{AD \, . \, BC} = -\frac{AC}{CB}\Big/\frac{AD}{DB} = -\{AB, CD\},$$

$$\therefore \{AC, BD\} = 1 - \{AB, CD\}$$
$$= 1 - \lambda.$$

Interchanging the first pair of $\{AC, BD\}$,

$$\{CA, BD\} = \frac{1}{1-\lambda}.$$

Again, as before,

$$\{BC, AD\} = 1 - \{BA, CD\}$$
$$= 1 - \frac{1}{\lambda}$$
$$= \frac{\lambda - 1}{\lambda},$$

$$\text{and } \{CB, AD\} = \frac{\lambda}{\lambda - 1}.$$

We thus have six different cross-ratios,

$$\lambda, \ \frac{1}{\lambda}, \ 1-\lambda, \ \frac{1}{1-\lambda}, \ \frac{\lambda-1}{\lambda}, \ \frac{\lambda}{\lambda-1}.$$

As a matter of fact, there is seldom any need to consider these various cross-ratios; it is customary to use the same cross-ratio throughout a given calculation, and it does not often become necessary to define which of the six possible cross-ratios is being used. The comma therefore is generally omitted, and the cross-ratio written $\{ABCD\}$.

Definition. If OA, OB, OC, OD be a pencil of four lines, the **cross-ratio of the pencil** is defined to be

$$\frac{\sin AOC}{\sin COB} : \frac{\sin AOD}{\sin DOB},$$

the sense of angles being taken into account (see p. 5); the cross-ratio of the pencil is written $O\{AB, CD\}$.

It is important to notice that the cross-ratio is unaltered if we substitute for any ray of the pencil (say OB) its prolongation backwards through O (say OB′).

For $\angle$ COB′ $= \angle$COB $+ 180° + n \,.\, 360°$,

$\angle$ DOB′ $= \angle$ DOB $+ 180° + n \,.\, 360°$.

sin COB′ $= -$ sin COB, sin DOB′ $= -$ sin DOB.

The cross-ratio is therefore unaltered. In fact, the cross-ratio pertains to the four *complete rays*, not to the four *half-rays*.

Ex. **527.** In Th. 28 it was shown that a system of two lines and the bisectors of the angles between them is a particular case of a harmonic pencil. Prove that the cross-ratio of such a pencil, as given by the sine definition, is equal to -1.

The cross-ratios of ranges and pencils are brought into relation by the following fundamental theorem.

Theorem 50.

The cross-ratio of a pencil is equal to the cross-ratio of the range in which any transversal cuts that pencil.

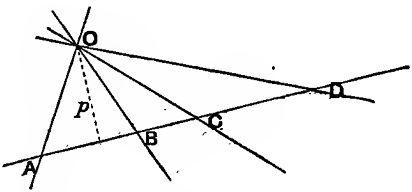

fig. 71.

To prove that O {AB, CD} = {AB, CD}.

I. *As regards sign.*

$$\frac{\sin \mathrm{AOC}}{\sin \mathrm{COB}} \text{ has the same sign as } \frac{\mathrm{AC}}{\mathrm{CB}};$$

$$\frac{\sin \mathrm{AOD}}{\sin \mathrm{DOB}} \text{ has the same sign as } \frac{\mathrm{AD}}{\mathrm{DB}}.$$

$\therefore$ O {AB, CD} has the same sign as {AB, CD}.

II. *As regards magnitude.*

Draw p the perpendicular from O upon ABCD.

$$\triangle AOC = \tfrac{1}{2} OA . OC \sin AOC,$$

$$\triangle COB = \tfrac{1}{2} OC . OB \sin COB,$$

$$\triangle AOD = \tfrac{1}{2} OA . OD \sin AOD,$$

$$\triangle DOB = \tfrac{1}{2} OD . OB \sin DOB.$$

$$\therefore \frac{\triangle AOC}{\triangle COB} : \frac{\triangle AOD}{\triangle DOB} = \frac{\sin AOC}{\sin COB} : \frac{\sin AOD}{\sin DOB}$$

$$= O\{AB,\ CD\}.$$

Again, $\triangle AOC = \tfrac{1}{2} p . AC,$

$$\triangle COB = \tfrac{1}{2} p . CB,$$

etc.

$$\therefore \frac{\triangle AOC}{\triangle COB} : \frac{\triangle AOD}{\triangle DOB} = \frac{AC}{CB} : \frac{AD}{DB}$$

$$= \{AB,\ CD\}.$$

$$\therefore O\{AB,\ CD\} = \{AB,\ CD\}.$$

Theorem 51.

If two lines cut a pencil in the ranges ABCD, A′B′C′D′, then {ABCD} = {A′B′C′D′}.

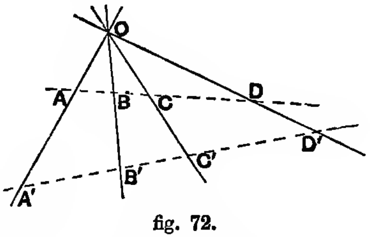

fig. 72.

For both {ABCD} and {A′B′C′D′} are equal to O {ABCD}.

Ex. **528.** Verify graphically the truth of Th. 51.

Ex. **529.** Prove that, while the cross-ratios of the ranges ABCD, A′B′C′D′ are equal, the ratios themselves (AB : BC, A′B′ : B′C′, etc.) are not equal unless (I) the two lines meet at infinity, or (2) O is at infinity.

Ex. **530.** If a transversal be drawn parallel to the ray OD of a pencil O {ABCD}, and cut the rays OA, OB, OC in P, Q, R respectively, then

$$PQ : RQ = O\,\{AC, BD\}.$$

Theorem 52.

If two pencils are subtended by the same range, then the cross-ratios of the pencils are equal.

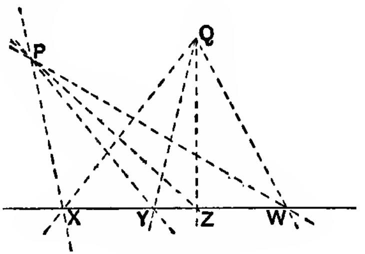

fig. 73.

For both P {XYZW} and Q {XYZW} are equal to {XYZW}.

Ex. **531.** Verify graphically the truth of Th. 52.

Ex. **532.** Examine what becomes of Th. 52, if

(i) P and Q are at infinity,

(ii) XYZW is the line at infinity.

Ex. **533.** Show that the two pencils subtended at points P, Q by the same range XYZW cannot be *equiangular* unless XYZW is the line at infinity.

(It may be noted that, if XYZW is the line at infinity, PQXYZW are concyclic, as a straight line together with the line at infinity is a limiting form of a circle. But, if PQXYZW are concyclic, ∠ XPY = ∠ XQY, etc.)

Ex. **534.** Consider the pencil D $\{AYCZ\}$ in fig. 74:

(i) It is cut by AB; what range on AB is equicross with $\{AYCZ\}$?

(ii) What range on XE is equicross with $\{AYCZ\}$?

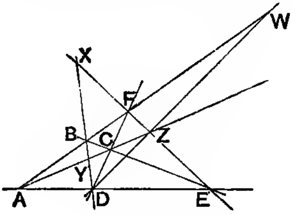

fig. 74.

Cross-ratio of a pencil of parallel lines.

If the vertex of a pencil retreats to infinity, the rays become parallel, and the angles of the pencil become zero. By the principle of continuity, we may be assured that all transversals still cut the pencil in equicross ranges; this property is, however, obvious from the fact that any two transversals are divided similarly by a pencil of parallel lines.

The angles of the pencil being zero, it would not appear, at first sight, that the ordinary definition of the cross-ratio of a pencil has no application to this case. This difficulty may be avoided by defining the cross-ratio of a pencil of parallel lines as the cross-ratio of the range in which any transversal cuts the pencil.

We may use the property $\underset{\theta=0}{\mathrm{Lt}} \dfrac{\sin\theta}{\theta}=1$ to illustrate the case of a pencil of parallel lines. For suppose that a circle be drawn with centre O so that the pencil intercepts arcs AB, BC, CD.

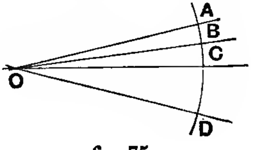

fig. 75.

As O retreats towards infinity, let the radius be increased and the angles be diminished in such a way that the arcs remain finite.

Then
$$\text{Lt}\ \frac{\sin AOC}{\sin COB}\Big/\frac{\sin AOD}{\sin DOB}=\frac{\angle AOC}{\angle COB}\Big/\frac{\angle AOD}{\angle DOB}$$
$$=\frac{\text{arc } AC}{\text{arc } CB}\Big/\frac{\text{arc } AD}{\text{arc } DB},$$

and ultimately the ratios of the arcs become the ratios of the segments of a transversal line.

THEOREM 53.

If $\{ABCD\}$, $\{A'B'C'D'\}$ be two equicross ranges, and if AA', BB', CC' be concurrent, then DD' must pass through the point of concurrence.

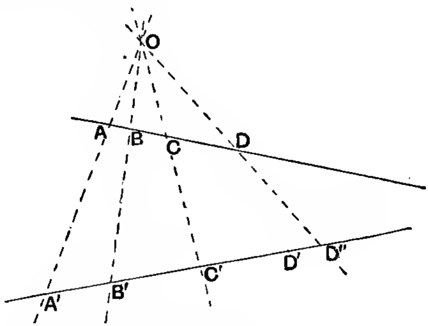

fig. 76.

Let O be the point of concurrence of AA', BB', CC'.

If DD' does not pass through O, let OD cut $A'B'$ in D''.

Then
$$\{A'B', C'D''\}=\{AB, CD\}$$
$$=\{A'B', C'D'\}.$$
$$\therefore \frac{A'C' . D''B'}{C'B' . A'D''}=\frac{A'C' . D'B'}{C'B' . A'D'},$$
$$\therefore \frac{D''B'}{A'D''}=\frac{D'B'}{A'D'},$$

$\therefore$ D'' coincides with D',

$\therefore$ DD' passes through O.

Note. This theorem and Theorem 51 could be stated as theorem and converse. It must be carefully noted that it is generally *not* true that, if $\{ABCD\} = \{A'B'C'D'\}$, then AA', BB', CC', DD' are concurrent.

Ex. **535.** Examine the particular case in which $\{ABCD\}$, $\{A'B'C'D'\}$ are similar.

Ex. **536.** Place two *similar* ranges $\{ABCD\}$, $\{A'B'C'D'\}$ in such a position that AA', BB', CC', DD' are *not* concurrent.

Theorem 54.

If two equicross ranges $\{PXYZ\}$, $\{PX'Y'Z'\}$ have a point P in common, then XX', YY', ZZ' are concurrent.

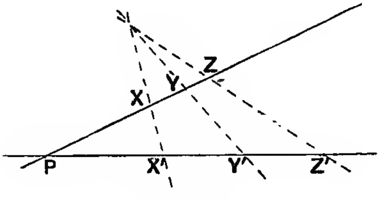

fig. 77.

This is a particular case of Theorem 53.

Ex. **537.** Prove this theorem without assuming Th. 53.

THEOREM 55.

If P {XYZW}, Q {XYZW} be two equicross pencils, and if X, Y, Z be collinear, then W is on the line XYZ.

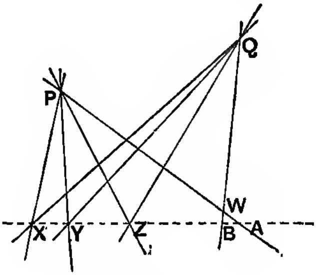

fig. 78.

If W does not lie on XYZ, let PW, QW cut XYZ in A, B respectively.

Then

$$\{XY, ZA\} = P\{XY, ZW\}$$

$$= Q\{XY, ZW\}$$

$$= \{XY, ZB\}.$$

$$\therefore \frac{XZ \,.\, AY}{ZY \,.\, XA} = \frac{XZ \,.\, BY}{ZY \,.\, XB},$$

$$\therefore \frac{AY}{XA} = \frac{BY}{XB},$$

$\therefore$ A and B coincide,

$\therefore$ W lies on XYZ.

THEOREM 56.

If two equicross pencils P {ABCD}, Q {ABCD} have a ray PQA in common, then BCD are collinear.

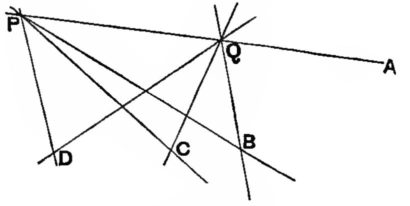

fig. 79.

This is a particular case of Theorem 55.

Ex. **538.** Prove Th. 56 without assuming Th. 55.

Ex. **539.** Prove that in fig. 77 the intersections of XY′, X′Y; of XZ′, X′Z; of YZ′, Y′Z lie on a line through P. (Consider two of the above points.)

Ex. **540.** Join the intersection of QB, PC to that of QC, PB; that of QB, PD to that of QD, PB; that of QC, PD to that of QD, PC. Prove that these three lines meet on PQ.

CROSS-RATIOS AND PROJECTION.

We have seen that angle properties as a rule are destroyed by orthogonal projection. One important set of angle relations, however, are undisturbed; namely, those connected with cross-ratios. The reader will be able to appreciate the importance of cross-ratio in view of the following theorems.

A range of points is equicross with the range obtained by projecting these points.

A pencil of lines is equicross with the pencil obtained by projecting these lines.

The proofs are left to the reader.

It follows from the above theorems that harmonic properties of points and lines are unaltered by projection.

EXERCISES ON CHAPTER XIII.

Ex. **541.** Find a point on a given line such that if it be joined to three given points in a plane with the line, any parallel to the line is divided in a given ratio by the three joins.

Ex. **542.** **Four fixed points on a circle subtend at a variable point on the circle a pencil of constant cross-ratio.**

Ex. **543.** **Four fixed tangents to a circle meet a variable tangent to the circle in a range of constant cross-ratio.**

(Consider the pencil subtended at the centre.)

Ex. **544.** **If four points are collinear, their polars with respect to a circle are concurrent; the cross-ratio of the pencil so formed is equal to that of the range formed by the four points.**

Ex. **545.** X is the vertex of a fixed angle; PAB is a transversal which turns about a fixed point P and cuts the arms of the angle in A, B; O, O′ are two fixed points collinear with X. OA, O′B meet in Q. Prove that the locus of Q is a straight line.

(Consider a pencil formed by PX and three positions of the transversals PA_1B_1, PA_2B_2, PA_3B_3.)

Ex. **546.** With the notation of the preceding exercise, let O, O′ be collinear with P instead of X. Prove that the locus of Q is a straight line through X.

(Consider a pencil formed by POO′ and three positions of the transversal.)

Ex. **547.** Prove that if the sides of the triangle $O_1O_2O_3$ pass through the vertices of the triangle $U_1U_2U_3$, and A_1 be any point on U_2U_3, and O_3A_1 meet U_1U_3 in A_2, and O_2A_1 meet U_1U_2 in A_3, then O_1, A_2, A_3 are collinear.

(Consider pencils whose vertices are A_1, U_1.)

Ex. **548.** Three points F, G, H are taken on the side BC of a triangle ABC; through G any line is drawn cutting AB and AC in L and M respectively; FL and HM intersect in K; prove that K lies on a fixed straight line passing through A.

Ex. **549.** The three sides of a varying triangle ABC pass each through one of three fixed collinear points P, Q, R. Further, A and B move along fixed lines; show that C also moves on a fixed line, concurrent with the other two.

Ex. **550.** A straight line drawn through a point P meets two fixed straight lines in the points L and M. The straight lines joining L and M to a point Q meet the fixed straight lines again in the points M′ and L′. Show that if P and Q are fixed, L′M′ passes through a fixed point.

Ex. **551.** Show that the lines joining the centres of the escribed circles of a triangle to the corresponding vertices of the pedal triangle are concurrent.

Ex. **552.** Prove that the lines joining the centres of the escribed circles of a triangle to the middle points of the corresponding sides are concurrent.

Ex. **553.** A′, B′, C′ are the mid-points of the sides of the triangle ABC, and any line is drawn to meet the sides of the triangle A′B′C′ in K, L, M. AK, BL, CM meet the sides of ABC in K′, L′, M′ respectively. Prove that K′L′M′ is a straight line.

Ex. **554.** If A′, B′, C′ be three points on the sides of a triangle ABC such that $AB' \cdot BC' \cdot CA' = AC' \cdot BA' \cdot CB'$ and X, Y, Z be the mid-points of B′C′, C′A′, A′B′, then AX, BY, CZ are concurrent.

Ex. **555.** Two points X, Y separate harmonically each of the three pairs of points P and P′, Q and Q′, R and R′. Prove that

$$\{PP'QR\} = \{P'PQ'R'\}.$$

CHAPTER XIV.

THE PRINCIPLE OF DUALITY.

THE COMPLETE QUADRILATERAL AND QUADRANGLE.

The reader may have noticed that there exists in plane geometry a certain **duality**, by which many properties of *points* have, as their counterpart, corresponding properties of *lines*.

For instance :—

2 points define 1 line.	2 lines define 1 point.
3 points define 3 lines.	3 lines define 3 points.
4 points define 6 lines.	4 lines define 6 points.
etc.	etc.
A point moving under certain conditions defines a curve, the *locus*.	A line moving under certain conditions defines a curve, the *envelope*.
If a point lies in a fixed line, its polar with respect to a circle passes through a fixed point.	If a straight line passes through a fixed point, its pole with respect to a circle lies in a fixed line.

This duality has obvious limitations, though a more extended study of geometry will show that it reaches further than would appear at first sight: e.g. there would at first sight seem to be no point-system corresponding with a line-system of two lines at right angles.

However, there are many cases of duality that may be cited at this stage.

In order to exhibit the matter in the most striking way, it is convenient to use two new terms:—

Definition. The **join** of two points is the unlimited line defined by the two points.

Definition. The **meet** of two lines is the point defined by the two lines (by their intersection).

It is also convenient to denote points by large letters, and lines by small letters: AB is the join of points A, B; *ab* is the meet of lines *a*, *b*.

Using this notation:—

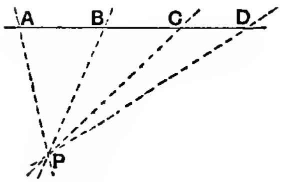

fig. 80.

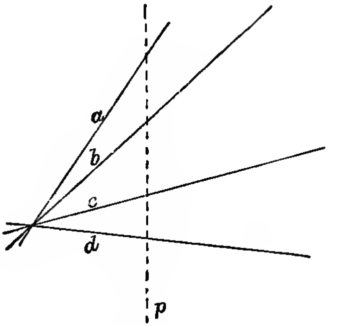

fig. 81.

A range of four fixed points A, B, C, D together with a varying point P define a pencil of constant cross-ratio.

If two equicross ranges ABCD, A′B′C′D′ be placed so that the lines AA′, BB′, CC′ are concurrent, then DD′ will be concurrent with these three lines.

A pencil of four fixed lines *a*, *b*, *c*, *d* together with a varying line *p* define a range of constant cross-ratio.

If two equicross pencils *abcd*, *a′b′c′d′* be placed so that the points *aa′*, *bb′*, *cc′* are collinear, then *dd′* will be collinear with these three points.

An interesting case is that of the complete figures defined by four lines and four points.

Definitions.

Four lines together with their six meets form a **complete quadrilateral** (or four-line).

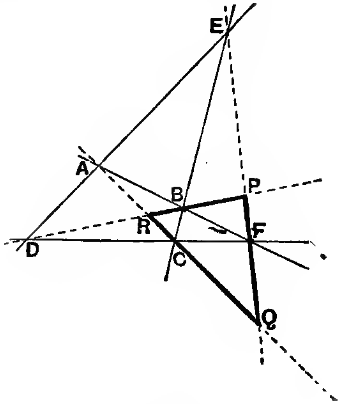

fig. 82.

The four lines AB, BC, CD, DA are called **sides**.

The meet of any two sides is called a **vertex**; the vertices are the six points A, B, C, D, E, F.

Opposite vertices are vertices that do not lie on the same side (A, C; B, D; E, F).

The join of two opposite vertices is called a **diagonal**; these are three in number, AC, BD, EF.

Four points together with their six joins form a **complete quadrangle** (or four-point).

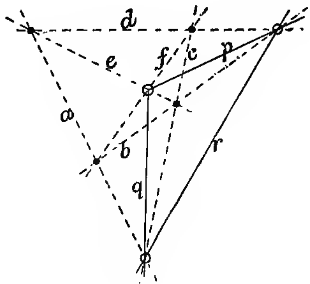

fig. 83.

The four points *ab*, *bc*, *cd*, *da* are called **vertices**.

The join of any two vertices is called a **side**; the sides are the six lines *a*, *b*, *c*, *d*, *e*, *f*.

Opposite sides are sides that do not pass through the same vertex (*a*, *c*; *b*, *d*; *e*, *f*).

The meet of two opposite sides is called a **diagonal-point**; these are three in number, *ac*, *bd*, *ef*.

We will now prove the important harmonic property of the complete $\begin{cases}\text{quadrilateral}\\ \text{quadrangle}\end{cases}$.

Before proving this, it should be noted that

If ABCD is a range of points and P a point not lying on the same line, P {ABCD} signifies the cross-ratio of the pencil PA, PB, PC, PD.

If *abcd* is a pencil of lines and *p* a line not passing through the same point, *p* {*abcd*} signifies the cross-ratio of the range *pa*, *pb*, *pc*, *pd*.

Theorem 57.

In a complete quadrilateral, on each diagonal there is a harmonic range formed by its meets with the other two diagonals together with two vertices of the quadrilateral.

To prove {EF, PQ} a harmonic range.

$$\{EF, PQ\} = B\{EF, PQ\}$$
$$= \{CA, RQ\}$$
$$= D\{CA, RQ\}$$
$$= \{FE, PQ\}^*.$$

Since the points E, F have been interchanged without altering the value of the cross-ratio, {EF, PQ} is harmonic.

Theorem 58.

In a complete quadrangle, through each diagonal-point there is a harmonic pencil formed by its joins to the other two diagonal-points together with two sides of the quadrangle.

To prove {*ef*, *pq*} a harmonic range.

$$\{ef, pq\} = b\{ef, pq\}$$
$$= \{ca, rq\}$$
$$= d\{ca, rq\}$$
$$= \{fe, pq\}.$$

Since the lines *e*, *f* have been interchanged without altering the value of the cross-ratio, {*ef*, *pq*} is harmonic.

* This method of proof may be remembered as follows: the range on diagonal 1 is projected on to diagonal 2, and back again on to diagonal 1; using the two vertices that lie in diagonal 3.

The above proof is of interest as bringing out the principle of duality. The following proof, however, may be preferred for ordinary purposes.

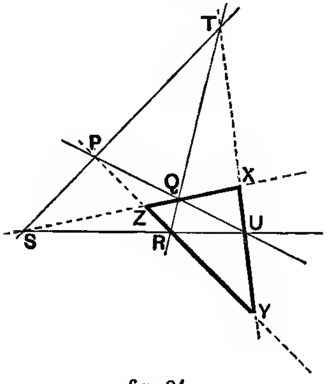

fig. 84.

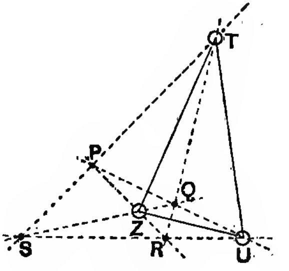

fig. 85.

Fig. 84 represents a complete quadrilateral.

To prove that {TU, XY} is a harmonic range.

Fig. 85 represents a complete quadrangle.

To prove that Z {QR, TU} is a harmonic pencil.

Consider the triangle STU.

Since SX, TR, UP are concurrent,

$$\therefore \frac{TX}{UX} \cdot \frac{UR}{SR} \cdot \frac{SP}{TP} = -1. \qquad [\textit{Ceva.}$$

Again, since P, R, Y are collinear,

$$\therefore \frac{TY}{UY} \cdot \frac{UR}{SR} \cdot \frac{SP}{TP} = 1. \qquad [\textit{Menelaus.}$$

$$\therefore \frac{TX}{UX} = -\frac{TY}{UY}.$$

∴ {TU, XY} is a harmonic range.

Hence Z {TU, QR} is a harmonic pencil.

Ex. **556.** Prove the above theorem for the other two diagonals of the quadrilateral, and for the other two diagonal points of the quadrangle.

Ex. **557.** AB is parallel to DC; AC, BD meet in Q; DA, CB in P. Prove that PQ bisects AB and DC.

Ruler construction for the fourth point of a harmonic range.

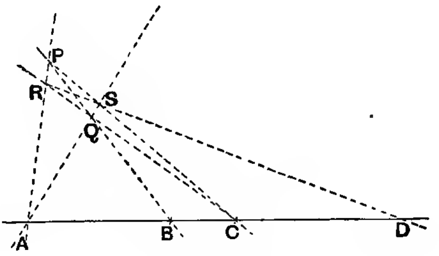

fig. 86.

Given three collinear points A, B, C; to find the point D such that {AC, BD} shall be harmonic.

Through A draw any two lines AP, AQ.

Through B draw any line BQP cutting the two former lines in Q and P respectively.

Join CQ, CP. Let these joins cut AP, AQ in R, S respectively.

Join RS and produce it to meet ABC in D.

Then, by the harmonic property of the quadrilateral PRQS, {AC, BD} is harmonic.

Note. This ruler construction for a fourth harmonic point is important, as showing that the idea of a harmonic range can be developed without any reference to **measurements** of lines or angle; in other words, can be put on a **non-metrical** or **projective** basis.

Ex. **558.** Perform the above construction for the point D, satisfying yourself that the same point is obtained however the lines AP, AQ, BQP are varied.

Ex. **559.** Bisect a line AC by the above method. [D will be at infinity.]

Ex. **560.** Show that, if one diagonal of a complete quadrilateral is parallel to the third (the exterior) diagonal, then the second diagonal bisects the third.

Ex. **561.** Apply the harmonic property of the quadrilateral to the case of the parallelogram, considering all three diagonals.

Self-polar Triangle.

The reader is reminded of the following theorems proved in Chapter VII.

Th. 31. **If a straight line is drawn through any point to cut a circle, the line is divided harmonically by the circle, the point, and the polar of the point with respect to the circle.**

Th. 32. **If the polar of a point P with respect to a circle passes through a point Q, then the polar of Q passes through P.**

Th. 33. **Two tangents are drawn to a circle from a point A on the polar of a point B; a harmonic pencil is formed by the two tangents from A, the polar of B and the line AB.**

Ex. **562.** Let the polars of points A, B, C form a triangle PQR. Prove that the polars of the points P, Q, R are the sides of the triangle ABC.

Ex. **563.** Draw the polar of a point A. On this polar take a point B. Draw the polar of B, passing through A (why?) and cutting the polar of A in C. Prove that AB is the polar of C; i.e. that **each side of the triangle ABC is the polar of the opposite vertex.**

Definition. If a triangle be such that each side is the polar of the opposite vertex with respect to a given circle, the triangle is said to be **self-polar** or **self-conjugate** with respect to the circle; and the circle is said to be **polar** with respect to the triangle.

From Ex. 563 it is seen that an **infinite number of triangles** may be drawn self-polar with respect to a given circle. One vertex may be taken anywhere in the plane; the second is then limited to a certain line; and when the second is fixed, the third is thereby fixed.

On the other hand, it will appear from Exs. 564, 565, that a given triangle has **only one polar circle.**

Ex. **564.** The centre of a circle, polar with respect to a given triangle, is the orthocentre of the triangle.

Ex. **565.** If H be the orthocentre of $\triangle$ABC, and AD, BE, CF the altitudes, then

$$HA \cdot HD = (\text{rad. of polar circle})^2,$$

and similarly

$$HB \cdot HE = (\text{rad. of polar circle})^2 = HC \cdot HF,$$

the sense of lines being taken into account.

Ex. **566.** A triangle self-polar with respect to a real circle cannot be acute-angled.

Ex. **567.** What is the polar circle of a right-angled triangle?

Ex. **568.** An isosceles triangle ABC has base $2a$ and vertical angle (A) 120°. Show that the radius of the polar circle is $a\sqrt{2}$. If the polar circle cuts AC in P, show that $\angle$ABP$=15°$.

Ex. **569.** **The sides of a triangle are divided harmonically by its polar circle.**

Ex. **570.** A triangle self-polar with respect to a point-circle is right-angled.

Ex. **571.** What does a self-polar triangle become if one vertex coincides with the centre of the circle?

Ex. **572.** If a circle consists of a straight line and the line at infinity, what do its self-polar triangles become?

Ex. 573. The angle A of a triangle ABC is obtuse; AD, BE, CF are the altitudes; H the orthocentre. The polar circle cuts AC in P and Q. Show that $EP^2 = EA . EC$, and that H, F, P, D, B, Q are concyclic.

Ex. 574. If circles are described in the sides of a triangle as diameters, they are cut orthogonally by the polar circle of the triangle.

Theorem 59.

If a quadrangle be inscribed in a circle, the triangle formed by the diagonal points is self-polar with respect to the circle.

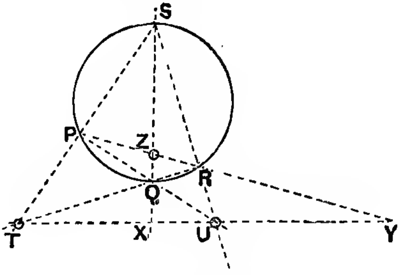

fig. 87.

We will prove that the side TU of the triangle TUZ is the polar of the vertex Z.

By Theorem 58 T {ZU, SQ} is a harmonic pencil.

∴ the pencil is cut by SQ in the harmonic range {ZX, SQ}.

∴ X is on the polar of Z. Th. 31.

Again, T {ZU, SQ} is cut by PR in the harmonic range {ZY, PR}.

∴ Y is on the polar of Z. Th. 31.

∴ XY or TU is the polar of Z.

Similarly it may be shown that UZ is the polar of T and ZT the polar of U.

Ex. 575. Prove in detail that UZ is the polar of T, and ZT the polar of U.

THEOREM 60.

If a quadrilateral be circumscribed about a circle, the triangle formed by the diagonals is self-polar with respect to the circle.

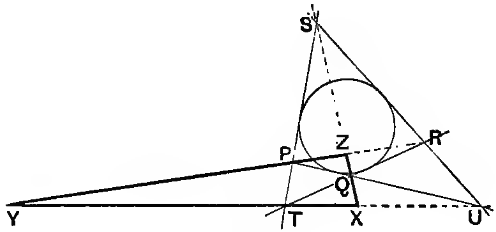

fig. 88.

We will prove that the vertex Z of the triangle XYZ is the pole of XY.

By Theorem 57, {XZ, QS} is a harmonic range.

∴ U {XZ, QS} is a harmonic pencil.

∴ UZ passes through the pole of UX. Th. 33.

Again, T {XZ, QS} is a harmonic pencil

∴ TZ passes through the pole of TX. Th. 33.

∴ Z is the pole of XY.

Similarly it may be shown that X is the pole of YZ, and Y the pole of ZX.

Ex. **576.** Prove in detail that X is the pole of YZ, and Y the pole of ZX.

Triangles in perspective.

Definition. Two figures are said to be **in perspective** if the joins of corresponding pairs of points are all concurrent.

Theorem 61.

(Desargues' Theorem*.)

If two triangles are such that the lines joining their vertices in pairs are concurrent, then the intersections of corresponding sides are collinear.

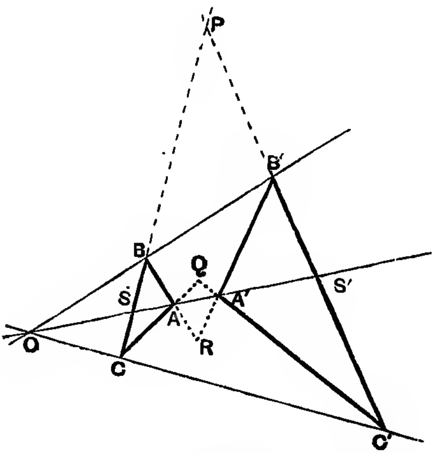

fig. 89.

The triangles ABC, A′B′C′ are such that AA′, BB′, CC′ meet at O.

Let BC, B′C′ meet at P; CA, C′A′ at Q; AB, A′B′ at R. Let OAA′ cut BC in S, B′C′ in S′.

* Gerard Desargues (born at Lyons, 1593; died, 1662).

To prove that PQR is a straight line.

$\{PBSC\} = \{PB'S'C'\}$ as both ranges lie on the pencil $O\{PBSC\}$.

$$\therefore \quad A\{PBSC\} = A'\{PB'S'C'\},$$

i.e.
$$A\{PROQ\} = A'\{PROQ\}.$$

These two equicross pencils, therefore, have a line OAA′ in common.

$\therefore$ P, Q, R are collinear. Th. 56.

Definition. The point O is called the **centre of perspective,** and the line PQR the **axis of perspective** of the two triangles ABC, A′B′C′ in fig. 89.

Ex. **577.** Prove Th. 61 by considering equicross pencils with vertices at B and B′ (instead of A and A′).

Ex. **578.** Investigate whether Th. 61 can be extended to the case of polygons in perspective.

Ex. **579.** Prove Th. 61 for the case in which the triangles ABC, A′B′C′ are not in the same plane. Hence prove the theorem for coplanar triangles by rotating the line OAA′ about O till it comes into the plane OBB′CC′.

Ex. **580.** Prove Th. 61 by considering fig. 89 as the representation *in plano* of three planes meeting at O and cut by the planes ABC, A′B′C′.

Ex. **581.** Prove the converse of Th. 61.

Ex. **582.** Prove that triangles that are similar and similarly situated (i.e. sides parallel) are in perspective. Where is the axis of perspective?

Ex. **583.** Investigate whether Ex. 582 can be extended to polygons.

Ex. **584.** Consider the case of triangles that are congruent and similarly situated.

Note on Three-dimensional Geometry.

The dual relation of point and line is confined to two-dimensional geometry.

In three dimensions, the point corresponds to the plane, the line occupying an intermediate position.

Thus:

Two points determine a line.	Two planes determine a line.
Three points determine a plane, unless they are all on the same line.	Three planes determine a point, unless they all contain the same line.
Two lines, in the same plane, determine a point.	Two lines, through the same point, determine a plane.
A point and a line determine a plane, unless the line passes through the point.	A plane and a line determine a point, unless the line lies in the plane.
etc.	etc.

Again, consider the five regular solids. They may be grouped as follows:

Tetrahedron (3 corners, 6 edges, 3 faces).

Cube (8C, 12E, 6F).	Octahedron (8F, 12E, 6C).
Dodecahedron (20C, 30E, 12F).	Icosahedron (20F, 30E, 12C).

The point-plane correspondence appears very clearly when we take stock of the cross-ratio properties of three dimensions.

We should begin with the definitions of:

(1) Cross-ratio of four points on the same line (a range of points).

(2) Cross-ratio of four planes containing the same line (a *sheaf* of planes), this being defined by means of the angles between the planes.

In addition there would be the definition of the

(3) Cross-ratio of four lines, in a plane, through a point (a pencil of lines).

There would then follow a number of theorems such as the following:

The joins of a point to the four points of a range give a pencil equicross with the range.

The intersections of a plane with the four planes of a sheaf give a pencil equicross with the sheaf.

The planes determined by a line and the four points of a range give a sheaf equicross with the range.

The points determined by a line and the four planes of a sheaf give a range equicross with the sheaf.

The proofs of the theorems may be left to the reader, who will find that these principles admit of further development*.

Exercises on Chapter XIV.

Ex. 585. A straight line meets the sides BC, CA, AB of a triangle in the points P, Q, R respectively; BQ and CR meet at X and AX meets BC at P′. Show that P and P′ are harmonic conjugates with respect to B and C.

If X is the orthocentre of ABC, show that XP is perpendicular to the straight line joining A to the middle point of BC.

Ex. **586.** The collinear points ADC are given; CE is any other fixed line through C, E is a fixed point and B is any moving point on CE. The lines AE, BD intersect in Q; the lines CQ, DE in R; and the lines BR, AC in P. Prove that P is a fixed point as B moves along CE.

Ex. **587.** If a line drawn through the intersection O of the diagonals of a quadrilateral cuts one pair of opposite sides in P, P′, so that OP=P′O, and cuts the other pair in Q, Q′, show that PQ=Q′P′.

* See Reye's *Geometry of Position*, translated by Holgate (the Macmillan Company).

Ex. **588.** Perpendiculars at B, C to the sides BA, CA of a triangle ABC meet the opposite sides in P, Q; and the tangents to the circumcircle at B, C meet in R. Prove that P, Q, R are collinear.

Ex. **589.** A quadrilateral is such that pairs of opposite sides have the same sum. If O be the orthocentre of the triangle formed by the diagonals, then O is also the in-centre of the quadrilateral.

Ex. **590.** Two tangents to a circle are fixed; two others are drawn so as to form with the two fixed tangents a quadrilateral having two opposite sides along the fixed tangents; show that the locus of the intersection of internal diagonals of this quadrilateral is a straight line, and find its position.

Ex. **591.** ABCD is a quadrilateral inscribed in a circle whose centre is O; AB, CD intersect in E; AD, BC intersect in F; AC, BD intersect in G. Prove that OG is perpendicular to EF; and that BC, AD subtend equal angles at the foot of the perpendicular from O upon EF.

Ex. **592.** Prove that the circle on each of the diagonals of a quadrilateral as diameter is orthogonal to the polar circle of each of the four triangles formed by the sides of the quadrilateral.

Ex. **593.** Prove that the mid-points of the diagonals of a complete quadrilateral are collinear.

(Let ABCDEF be the quadrilateral; EF being the third diagonal. Let P, Q, R be the mid-points of AC, BD, EF. Prove that $\triangle$s PQE, PQF are each $\frac{1}{4}$ of the quadrangle ABCD.)

Ex. **594.** ABC, A′B′C′, A″B″C″, are three triangles in perspective, and BC, B′C′, B″C″ are parallel. Prove that the line joining the intersections of AB, A′B′, and AC, A′C′, is parallel to the line joining the intersections of A′B′, A″B″, and A′C′, A″C″.

Ex. **595.** The lines EF, FD, DE which join the points of contact D, E, F of the inscribed circle of a triangle with the sides cut the opposite sides X, Y, Z. Prove that the mid-points of DX, EY, FZ are collinear.

Ex. **596.** Show that in a complete quadrangle the three sides of the harmonic triangle are met by the sides of the quadrangle in 6 points, other than the vertices of the harmonic triangle, which lie by threes on four straight lines.

MISCELLANEOUS EXERCISES.

Ex. **597.** ABC is a triangle; D, E, F are the feet of the perpendiculars. Prove that, if the triangles FBD, EDC are equal in area, AB is equal to AC.

Ex. **598.** In a given circle show how to inscribe a triangle ABC such that the angle ABC is given and the sides AB, AC pass through given points.

Ex. **599.** From a fixed point A straight lines ABC, AEF are drawn to meet two fixed lines in B, C and E, F. Prove that the circles circumscribing the triangles ABE, ACF intersect at a constant angle.

Ex. **600.** The perpendiculars drawn to the sides of a triangle at the points in which they are touched by the escribed circles are concurrent.

Ex. **601.** Three circles have two common points O and O′, and any straight line through O cuts them in points P, Q, and R. Prove that the circumscribing circle of the triangle formed by the tangents at P, Q, R passes through O′.

Ex. **602.** Draw a straight line from the vertex A of a triangle ABC meeting BC in P so that $AP^2 = BP \cdot CP$, considering the cases in which P (i) is, (ii) is not, situated between B and C.

Ex. **603.** A, B, C, D are four points in a plane: points P, Q, R are taken in AD, BD, CD respectively such that

$$AP : PD = BQ : QD = CR : RD.$$

Show that the three lines joining P, Q, R to the middle points of BC, CA, AB respectively are concurrent.

Ex. **604.** Prove that the locus of the middle points of the sides of all triangles which have a given orthocentre and are inscribed in a given circle is another circle.

Ex. **605.** A straight line drawn parallel to the median AD of an isosceles triangle ABC whose angle A is a right angle cuts the sidee AB, AC in P and Q. Show that the locus of M, the intersection of BQ, CP, is a circle; and that, if N is the middle point of PQ, MN touches this circle.

Ex. **606.** A straight line drawn through the vertex of a triangle ABC meets the lines DE, DF, which join the middle point D of the base to the middle points E, F of the sides, in X, Y; show that BY is parallel to CX.

Ex. **607.** The points of contact of the escribed circles with the sides BC, CA, AB produced when necessary, are respectively denoted by the letters D, E, F with suffixes 1, 2 or 3 according as they belong to the escribed circle opposite A, B or C. BE_2, CF_3 intersect at P; BE_1, CF_1 at Q; E_2F_3 and BC at X; F_3D_1 and CA at Y; D_1E_2 and AB at Z. Prove that the groups of points A, P, D_1, Q; and X, Y, Z; are respectively collinear.

Ex. **608.** The opposite sides of the hexagon ABCDEF are parallel, and the diagonal CF is parallel to the sides AB and DE; BC, AF intersect in P, CD, EF in Q, and BD, AE in R; show that P, Q, R are in one straight line.

Ex. **609.** Show that, if O be any point on the circumcircle of the triangle ABC, and OL be drawn parallel to BC to meet the circumcircle in L, then will LA be perpendicular to the pedal line of O with respect to the triangle.

Ex. **610.** ABC is a triangle inscribed in a circle, and tangents to the circle at A, B, C cut BC, CA, AB respectively in the points A′, B′, C′. Show that the middle points of AA′, BB′, CC′ lie on the radical axis of the circumcircle and nine-points circle.

Ex. **611.** If ABC is a triangle and DEF its pedal triangle, the perpendiculars from A, B, C upon EF, FD, DE respectively are concurrent.

Ex. **612.** ABC is a triangle right-angled at C. The bisector of the angle A meets BC in D, the circumcircle in G, and the perpendicular to AB through the circumcentre in F. Prove that 2FG = AD. Hence (or otherwise) construct a right-angled triangle, given the hypotenuse and the length of the line drawn bisecting one of the acute angles and terminated by the opposite side.

Ex. **613.** A point O is taken within an equilateral triangle ABC such that the angles AOB, BOC, COA are in the ratios 3 : 4 : 5. AD is drawn perpendicular to BC, and OD is joined. Show that each of the triangles into which ADC is divided by OA, OD, OC is similar to one of the triangles into which ABC is divided by OA, OB, OC.

Ex. **614.** Two circles intersect in the points B, D; a straight line ABC cuts the circles in A, C; AD, CD cut the circles again in P, Q; AQ, CP meet in R; prove that DPQR is a cyclic quadrilateral.

Ex. **615.** ABCD is a quadrilateral inscribed in a circle S. AC and BD meet in E, AB and DC in F. If a circle can be drawn to touch the sides of the quadrilateral FBEC, prove that its centre must lie on S.

Ex. **616.** A circle S passes through the centre of another circle S′; show that their common tangents touch S in points lying on a tangent to S′.

Ex. **617.** Through a fixed point O any straight line is drawn meeting two fixed parallel lines in P and Q. Through P and Q straight lines are drawn in fixed directions intersecting in R. Prove that the locus of R is a straight line.

Ex. **618.** A quadrilateral ABCD is inscribed in a circle, and through a point E on AB produced a straight line EFG is drawn parallel to CD and cutting CB, DA produced in F, G respectively. Show how to draw the circle that passes through F and G and touches the given circle.

Ex. **619.** In a triangle $A_1B_1C_1$ a circle is inscribed, touching the sides in $A_2B_2C_2$; and so on. Find the values of the angles of the triangle $A_nB_nC_n$, and give a construction for the directions of the sides when n is made infinite.

Ex. **620.** The lines WAX, XBY, YCZ, ZDW bisect the exterior angles of the convex quadrilateral ABCD. Show that an infinite number of quadrilaterals can be inscribed in XYZW whose sides are parallel respectively to the sides of ABCD, and whose perimeters are equal to the perimeter of ABCD.

Ex. **621.** A, B, C are three given points. Show how to describe a square having one vertex at A so that the sides opposite to A shall pass through B, C respectively.

Ex. **622.** Any point P is taken on the base BC of a triangle ABC, and a line PL parallel to BA meets AC in L, while a line PM parallel to CA meets AB in M. Show that the triangle PLM is a mean proportional between the triangles BMP, PLC.

Ex. **623.** OAB is a triangle. Any circle through A, B meets OA at P and OB at Q. PQ meets AB at X, PB meets AQ at Y. Find the locus of Y, and show that XY passes through a fixed point.

Ex. **624.** Prove that the radical axes of a fixed circle and the several circles of a coaxal system meet in a point. State the theorems which may be obtained by inverting this theorem with respect to (i) a limiting point, (ii) a point of intersection of the coaxal circles, (iii) any other point in the plane.

Ex. **625.** A trapezium ABCD has the opposite sides AB, CD parallel. Shew that the common chord of the circles described on the diagonals AC, BD as diameters is perpendicular to AB and CD, and concurrent with AD and BC.

Ex. **626.** Given three points A, B, C on a circle, determine geometrically a fourth point D on the circle, such that the rays PC, PD may be harmonic conjugates with respect to the rays PA, PB, where P is any point in the circle.

Show further that the intersection of AC, BD, that of AB, CD, that of the tangents at A and D, and that of the tangents at B and C are collinear.

Ex. **627.** Find the locus of the centre of a circle which bisects the circumferences of two given circles.

Ex. **628.** O is the radical centre of three circles. Points A, B, C are taken on the radical axes and AB, BC, CA are drawn. Prove that the six points in which these meet the three given circles lie on a circle.

If radii vectores are drawn from O to these six points they meet the three given circles in six points on a circle and its common chords with the three circles meet in pairs on OA, OB, OC.

Ex. **629.** On a given chord AB of a circle a fixed point C is taken, and another chord EF is drawn so that the lines AF, BE and the line joining C to the middle point of EF meet in a point O; show that the locus of O is a circle.

Ex. **630.** If O be the centroid of the n points A, B, C,... and if P be any variable point, then $AP^2 + BP^2 + CP^2 + \ldots = n \,.\, OP^2 + \text{constant}$.

If ABC... be a regular polygon inscribed in a circle, O the centre, and P any point on the circumference of this circle, then the centroid of the feet of the perpendiculars from P on OA, OB, OC, ... will lie on a fixed circle.

Ex. **631.** If A, B, C are three collinear points and P is any point whatever, prove that $BC \,.\, PA^2 + CA \,.\, PB^2 + AB \,.\, PC^2 = -BC \,.\, CA \,.\, AB$. Find the radius of the circle which touches the circles described on AB, BC, AC as diameters.

Ex. **632.** Prove that the tangents to the circumcircle of the triangle ABC at the vertices meet the opposite sides in collinear points.

Ex. **633.** If L, L′ are the limiting points of a family of coaxal circles, prove that any circle through L, L′ cuts the family orthogonally, and that if PP′ is a diameter of this circle, then the polars of P with respect to the family pass through P′.

Ex. **634.** A line drawn through L, a limiting point of a coaxal system of circles, cuts one of the circles at A and B. The tangents at A and B cut another circle of the system at P, Q and R, S respectively. Shew that PR and QS subtend equal angles at L.

Ex. **635.** P, Q are any two points; PM is drawn perpendicular to the polar of Q with respect to a circle, and QN is drawn perpendicular to the polar of P; if O is the centre of the circle, prove that

$$PM : QN = OP : OQ.$$

Ex. **636.** If P be the extremity of the diameter CP of any circle through L, C, where L, L′, C, C′ are the limiting points and centres of two fixed circles and L lies within the circle with C as centre, then the polar of P with regard to the circle with C′ as centre passes through a fixed point.

Ex. **637.** A chord of a fixed circle is such that the sum of the squares of the tangents from its extremities to another fixed circle is constant; prove that the locus of its middle point is a straight line.

Ex. **638.** A circle touches two given circles in P and P′, and intersects their radical axis in Q and Q′. Prove that PP′ passes through one of the centres of similitude of the given circles, and that the tangents at Q and Q′ are parallel to a pair of common tangents of the given circles.

Ex. **639.** State (without proof) the chief properties of any geometrical figure which persist after inversion. If Q, Q′ are inverse points with respect to a circle B, and R, R′ are the inverse points of Q, Q′ with respect to an orthogonal circle C, prove that R, R′ are inverse points with respect to the circle B.

Ex. **640.** Two circles intersect in A and B, and a variable point P on one circle is joined to A and B, and the joining lines, produced if necessary, meet the second circle in Q and R. Prove that the locus of the centre of the circle circumscribing PQR is a circle.

Ex. **641.** Two squares have a common angular point at A and their angular points taken in order the same way round are respectively A, B, C, D and A, B′, C′, D′. Prove that the lines BB′, CC′, and DD′ are concurrent.

Ex. **642.** A, B, A′, B′ are given points, and PQ is a given straight line. Find points C, C′ in PQ such that the area of the triangles ABC, A′B′C′ shall be equal, and CC′ shall be of a given length.

Ex. **643.** The middle points of the sides of a plane polygon A are joined in order so as to form a second polygon B; prove that about this polygon B either an infinite number of polygons other than A, or no other can be circumscribed with their sides bisected at the corners of B, according as the number of sides is even or odd.

Ex. **644.** A circle is inscribed in a triangle ABC touching the sides at P, Q, R. Show that the diameter of the circle through P, the line QR, and the line joining A to the middle point of BC, are concurrent.

Ex. **645.** A common tangent touches two circles in P and Q respectively; show that P and Q are conjugate points* with regard to any coaxal circle.

Ex. **646.** If one pair of opposite vertices of a square is a pair of conjugate points with respect to a circle, so will be the other pair.

Ex. **647.** Having given two non-intersecting circles; draw the longest and the shortest straight line from one to the other, parallel to a given straight line.

Ex. **646.** POP', QOQ' are two chords of a fixed circle and O is a fixed point. Prove that the locus of the other intersection of the circles POQ, $P'OQ'$ is a second fixed circle.

Ex. **649.** The points Q and R lie on the straight line AC and the point V on the straight line AD; VQ meets the straight line AB in Z, and VR meets AB in Y: X is another point on AB: XQ meets AD in U, and XR meets AD in W. Prove that YU, ZW, AC are concurrent.

Ex. **650.** Three circles pass through a point O and their other intersections are A, B and C. A point D is taken on the circle OBC, E on OCA, F on OAB. Prove that if $ODEF$ are concyclic $AF.BD.CE=FB.DC.EA$.

Ex. **651.** A, B are two fixed points, and a variable circle through them cuts a fixed circle in C, D. Prove that the line joining the intersections of AC, BD and AD, BC passes through a fixed point.

Ex. **652.** Having given six points A, B, C, A', B', C' such that $A'B$ is parallel to AB', $B'C$ is parallel to BC', and $C'A$ is parallel to CA', prove that if $A'B'C'$ are collinear, ABC also are collinear.

Ex. **653.** The angles APB, AQB subtended at two variable points P, Q by two fixed points A, B differ by a constant angle, and the two ratios AP/BP, AQ/BQ are proportionals. Show that if P describes a circle, Q describes either a circle or a straight line.

Ex. **654.** Prove that the sum of the squares on the tangents from a pair of conjugate points to a circle is equal to the square on the distance between them.

* Two points are said to be **conjugate** with respect to a circle if the polar of each point passes through the other.

Ex. **655.** Prove that, if in a plane the ratio of the distances from two points be the same for each of three points A, B, and C, the two points are inverse points with regard to the circle ABC. Prove also that the line bisecting BC at right angles meets the lines BA and CA in two such points.

Ex. **656.** If a circle S touch the circumcircle of a triangle ABC at P, prove that the tangents to S from A, B, C are in the ratios AP : BP : CP. What does this result become when the radius of the circle S increases indefinitely?

Ex. **657.** PQ and RS are interior and exterior common tangents to two circles. The circles QSR and SRP cut PQ at p, q respectively; and the circles PQS, PQR cut RS at r, s respectively. Shew that circles will pass through Q, S, q, s and through P, R, p, r, and that the rectangle contained by their radii equals the rectangle contained by the radii of the original circles.

Ex. **658.** A triangle of given shape is inscribed in a given triangle. Shew that the locus of its centroid is in general six straight lines.

Ex. **659.** A circle U of constant radius is described, having its centre at any point of the circumference of a fixed circle whose centre is O; the variable circle U cuts another fixed circle V; Y is the foot of the perpendicular from O on the common chord of U and V. Prove that the locus of Y is a circle.

Ex. **660.** If two fixed circles be cut by a variable straight line in four points in a harmonic range, show that the product of the perpendiculars upon it from the centres of the circles is constant.

Ex. **661.** Through any point O in the plane of a triangle ABC is drawn a transversal, cutting the sides in P, Q, R. The lines OA, OB, OC are bisected in A′, B′, C′; and the segments QR, RP, PQ of the transversal are bisected in P′, Q′, R′.

Show that the three lines A′P′, B′Q′, C′R′ are concurrent.

Ex. **662.** The four points ABCD form a quadrilateral of which the diagonals AC, BD intersect in O, and A′, B′, C′, D′ are the inverse points with regard to O as origin of A, B, C, D respectively. Show that A′B′C′D′ is a quadrilateral having its angles supplementary to those of ABCD and that, if turned over, it may be placed in the plane so as to have sides and diagonals parallel to those of ABCD.

Ex. **663.** If from any point on the circumference of a circle perpendiculars are drawn to the four sides and to the diagonals of an inscribed quadrilateral, prove that the rectangle contained by the perpendiculars on either pair of opposite sides is equal to that contained by the perpendiculars on the diagonals.

Ex. **664.** If a system of circles be drawn so that each bisects the circumferences of two given circles, then the polars of a given point with respect to the system of circles will be concurrent.

Ex. **665.** A line is drawn cutting two non-intersecting circles; find a construction determining two points on this line such that each is the point of intersection of the polars of the other point with respect to the two circles.

Ex. **666.** If, on the sides BC, CD of a quadrilateral ABCD of which two opposite angles at B and D are equal (the other two opposite angles being unequal) points E and F be taken such that the areas of the triangles AED, AFB are equal, prove that the radical axis of the circles on BF, ED as diameters passes through A.

Ex. **667.** Two opposite sides of a quadrilateral inscribable in a circle lie along two given lines OX, OY and the intersection of the diagonals is given; show that the locus of the centres of the circles is a straight line.

Ex. **666.** Two circles intersect orthogonally at a point P, and O is any point on any circle which touches the two former circles at Q and Q′. Show that the angle of intersection of the circumcircles of the triangles OPQ, OPQ′ is half a right angle.

Ex. **669.** The triangles $A_1B_1C_1$, $A_2B_2C_2$ are reciprocal with respect to a given circle; B_2C_2, C_1A_1 intersect in P_1 and B_1C_1, C_2A_2 in P_2. Show that the radical axis of the circles which circumscribe the triangles $P_1A_1B_2$, $P_2A_2B_1$ passes through the centre of the given circle.

Ex. **670.** Show that if each of two pairs of opposite vertices of a quadrilateral is conjugate with regard to a circle the third pair is also; and that the circle is one of a coaxal system of which the line of collinearity of the middle points of the diagonals is the radical axis.

Ex. **671.** Three circles C_1, C_2 and C_3 are such that the chord of intersection of C_2 and C_3 passes through the centre of C_1, and the chord of intersection of C_3 and C_1 through the centre of C_2; show that the chord of intersection of C_1 and C_2 passes through the centre of C_3.

Ex. **672.** A system of spheres touch a plane P (on either side of the plane) at a point O. A plane Q, not passing through O, cuts P in the line l, touches two of the spheres in L_1 and L_2 respectively, and cuts the other spheres. Show that the system of circles in which Q cuts the spheres is coaxal, with L_1 and L_2 as limiting points and l as radical axis.

Ex. **673.** Show that the locus of a point at which two given portions of the same straight line subtend equal angles is a circle.

Ex. **674.** Two variable circles touch each of two fixed circles and each other; show that the locus of the point of contact of the variable circles is a circle.

Ex. **675.** A, B, C, D are four circles in a plane, each being external to the other three and touching two of them. Show that the four points of contact are concyclic.

Ex. **676.** Three circles meet in a point O. The common chord of the first and second passes through the centre of the third, and the common chord of the first and third passes through the centre of the second. Prove by inversion with respect to O that the common chord of the second and the third passes through the centre of the first.

Ex. **677.** AOB is a right-angled triangle, O is the right angle, and OL is the perpendicular to AB. On the other side of OB remote from A the square OBGF is described, and the line AG cuts OL in M. Prove that

$$\frac{1}{OM}=\frac{1}{AB}+\frac{1}{OL}.$$

Ex. **678.** If A, B are conjugate points with respect to a circle (see note to Ex. 645), then the tangent to the circle from O, the mid-point of AB, is equal to OA.

Ex. **679.** The sides BC, DA of the quadrilateral ABCD are cut by any line in the points K, L respectively. AC, BD meet in X; AK, BL meet in Y; CL, DK meet in Z and BC, AD meet in E. Prove that

$$X\{KCZD\}=\{EALD\}=X\{KCYD\},$$

and that XYZ is a straight line.

APPENDIX.

The Pentagon.

To divide a given straight line into two parts such that the square on the greater part may be equal to the rectangle contained by the whole line and the smaller part.

[*Analysis.* Let the whole line contain a units of length.

Let the ratio of the greater part to the whole line be $x:1$.

Then the greater part contains ax units; and the smaller $a-ax$ units.

The square on the greater part contains a^2x^2 units of area and the rectangle contained by the whole line and the smaller part contains $a(a-ax)$ units of area,

$$\therefore\ a^2x^2 = a^2 - a^2x,$$

$$\therefore\ x^2 = 1 - x,$$

$$\therefore\ x^2 + x - 1 = 0.$$

Solving this equation, we find

$$x = \pm\frac{\sqrt{5}}{2} - \frac{1}{2}.$$

For the present* we reject the lower sign, which would give a negative value for x; and we are left with

$$x = \frac{\sqrt{5}}{2} - \frac{1}{2} = 0{\cdot}618\ldots.]$$

* It will be seen below (p. 162) that a meaning can be found for the negative value of x.

In order to construct this length with ungraduated ruler and compass only, we proceed as follows :—

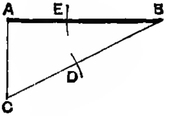

fig. 90.

Let AB be the given straight line.

Construction At A erect AC $\perp$ to AB, and equal to $\frac{1}{2}$AB.

Join CB.

From CB cut off CD = CA.

From BA cut off BE = BD.

Then AB is divided as required.

Proof
$$BC^2 = AB^2 + AC^2.$$

But AB $= a$ and AC $= \frac{1}{2}a$,

$$\therefore \ BC^2 = a^2 + \tfrac{1}{4}a^2$$
$$= \tfrac{5}{4}a^2.$$
$$\therefore \ BC = \sqrt{\tfrac{5}{4}}\,a = \frac{\sqrt{5}}{2}\,a.$$
$$\therefore \ BE = BD = \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)a.$$

To verify that this length satisfies the given conditions.

$$BE^2 = \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^2 a^2 = \left(\frac{5}{4} + \frac{1}{4} - \frac{\sqrt{5}}{2}\right)a^2$$
$$= \left(1\tfrac{1}{2} - \frac{\sqrt{5}}{2}\right)a^2.$$
$$AE = a - \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)a = \left(1\tfrac{1}{2} - \frac{\sqrt{5}}{2}\right)a.$$
$$\therefore \ AE \,.\, AB = \left(1\tfrac{1}{2} - \frac{\sqrt{5}}{2}\right)a \times a = BE^2.$$

Extreme and mean ratio. The relation AE . AB $=$ BE^2 may be written AE : BE $=$ BE : AB. Thus the straight line AB has been divided so that **the larger part is the mean proportional between the smaller part and the whole line.** In other words, the larger part is the mean, while the smaller part and the whole line are the extremes of a proportion. For this reason, a line divided as above is said to be **divided in extreme and mean ratio.** This method of dividing a line is also known as **medial section.**

NOTE. The solution $x = -\dfrac{\sqrt{5}}{2} - \dfrac{1}{2}$ was rejected. Strictly speaking, however, it is a second solution of the problem. The fact that this value of x is negative indicates that BE must be measured from B *in the other direction*—away from A rather than towards A—as BE′ in fig. 91.

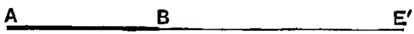

fig. 91.

Ex. **660.** With ruler and compass, divide a straight line one decimetre long in extreme and mean ratio. Calculate the correct lengths for the two parts, and estimate the percentage error in your drawing.

Ex. **661.** Devise a geometrical construction for dividing a line externally as in the above note (fig. 91).

Ex. **662.** Prove that, if E′ is constructed as in the note (fig. 91), then AB . AE′ $=$ BE′^{2}; and hence that the line AB is divided externally in extreme and mean ratio.

Ex. **663.** Prove that if AB is divided externally in extreme and mean ratio at E′, then AE′ is divided internally in extreme and mean ratio at B.

Ex. **664.** Divide a straight line AB at C so that

(i) AB . AC $=$ 2CB^2,

(ii) 2AB . AC $=$ CB^2,

(iii) AC^2 $=$ 2CB^2.

To construct an isosceles triangle such that each of the base angles is twice the vertical angle.

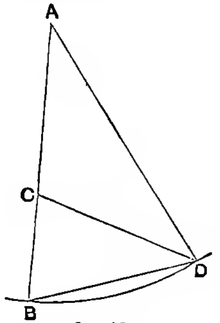

fig. 92.

Construction Draw a straight line AB of any length.

Divide AB at C so that $AB \cdot BC = AC^2$.

With centre A and radius AB describe a circle.

In this circle place a chord $BD = AC$.

Join AD.

Then ABD is an isosceles $\triangle$ having $\angle B = \angle D = 2 \angle A$.

Proof Join CD.

Since $BC \cdot BA = AC^2 = BD^2$,

$$\therefore \quad BC : BD = BD : BA.$$

Thus, in the $\triangle^s$ BCD, BDA, the $\angle B$ is common and the sides about the common angle are proportional.

$\therefore \quad \triangle^s$ are similar. IV. 5.

But $\triangle BDA$ is isosceles ($\because AB = AD$),

$\therefore \quad \triangle BCD$ is isosceles,

$\therefore \quad CD = BD = CA.$

$\therefore \quad \angle CDA = \angle A.$

Now $\angle BCD$ (ext. $\angle$ of $\triangle CAD$)

$= \angle A + \angle CDA$

$= 2 \angle A,$

$\therefore \quad \angle B = 2 \angle A.$

Ex. **685.** Perform the above construction. Calculate what should be the magnitudes of the angles of the triangle, and verify that your figure agrees with your calculation. (To save time, it will be best to divide AB in the required manner *arithmetically*, i.e. by measuring off the right length.)

Ex. **686.** Show that, in fig. 92, BD is the side of a regular decagon inscribed in the circle.

Ex. **687.** Show that, if ⊙ACD is drawn, BD will be a tangent to that circle.

Ex. **688.** Prove that AC and CD are sides of a regular pentagon inscribed in ⊙ACD.

Ex. **689.** Let DC be produced to meet the circle of fig. 92 in E. Prove that BE is the side of a regular 5-gon inscribed in ⊙A.

Ex. **690.** Prove that AE = EC. (See Ex. 689.)

Ex. **691.** Prove that AE is ∥ to BD. (See Ex. 689.)

Ex. **692.** Prove that △s AED, CAD are similar. (See Ex. 689.)

Ex. **693.** Prove that DE is divided in extreme and mean ratio at C. (See Ex. 689.)

Ex. **694.** Prove that, if ⊙ABD is drawn, BD is the side of a regular pentagon inscribed in the ⊙.

Ex. **695.** Let the bisectors of ∠s B, D meet ⊙ABD in F, G. Prove that AGBDF is a regular pentagon.

To describe a regular pentagon.

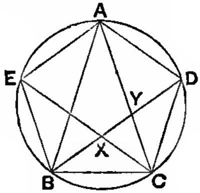

fig. 93.

Construction Construct an isosceles $\triangle$ ABC with each of its base angles twice the vertical angle.

Draw the circumscribing $\odot$ of $\triangle$ ABC.

Then BC is a side of a regular 5-gon inscribed in $\odot$ABC.

Proof Since $\angle$ ABC $= \angle$ ACB $= 2 \angle$ BAC,

$\therefore \quad \angle \text{BAC} = \frac{1}{5}$ of 2 rt. $\angle$ s $= 36°$.

$\therefore$ BC subtends $36°$ at the circumference and $72°$ at the centre.

$\therefore$ BC is a side of a regular 5-gon inscribed in the $\odot$.

The pentagon may now be completed. (How?)

Practical method of describing a regular pentagon.

The above method is interesting theoretically, but inconvenient in practice. The practical method is as follows.

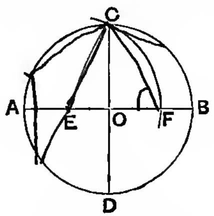

fig. 94.

Draw AOB, COD, two perpendicular diameters of a circle.

Bisect OA at E.

With centre E and radius EC describe a $\odot$ cutting OB in F.

Then CF is equal to a chord of a regular pentagon inscribed in the $\odot$ O.

(The proof of this needs some knowledge of Trigonometry.)

Ex. **696.** Prove that in fig. 93 AB, CE divide each other in extreme and mean ratio.

Ex. **697.** In fig. 93, show that $\triangle$ DCX is similar to $\triangle$ ABC.

Ex. **698.** Show that $\triangle$ CXY is similar to $\triangle$ ABC.

Ex. **699.** Prove that BY is divided in medial section at X.

Ex. **700.** Prove that BY is the mean proportional between BX and BD.

To prove that $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$.

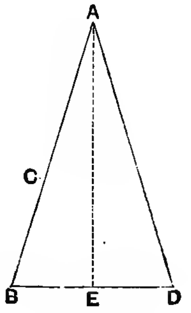

fig. 95.

Let ABD be an isosceles $\triangle$ having $\angle B = \angle D = 2\angle A$ (see page 163); let AC = BD as in fig. 92, and let AE be drawn to bisect BD at rt. $\angle$s.

Then AB is divided in extreme and mean ratio at C.

Thus, if $AB = a$, $AC = \dfrac{\sqrt{5}-1}{2}a$ (see p. 160).

$$\text{Now } \angle BAD = 36^\circ \text{ (p. 165)},$$

$$\therefore \angle BAE = 18^\circ,$$

$$\therefore \sin 18^\circ = \frac{BE}{AB} = \frac{BD}{2AB} = \frac{\sqrt{5}-1}{4}.$$

Ex. **701.** Calculate sin 18° as a decimal; and verify the value by measurement.

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